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ELEMENTARY ANALYTICAL STATICS

FOR

B.A. (PASS & HONOURS) STUDENTS
OF INDIAN UNIVERSITIES

by

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PREFACE TO THE THIRD EDITION.

The book has been thoroughly revised and some of the chapters have been re-arranged and re-written. The number of solved examples has been enlarged by a few more typical examples.

A chapter on Forces in Three Dimensions has been added to suit the convenience of students studying the Honours course of Delhi University.

DELHI :
September 1949.

S. R. GUPTA

PREFACE TO THE SECOND EDITION.

The book has been enlarged and thoroughly revised. The subject matter has been re-arranged and largely re-written and many sets of examples have been added. Chapter II has been split up into two, and one chapter has been separately devoted to the solution of problems.

The scope of the book has been still further enlarged by the addition of a discussion on the following topics :—

(1) The equilibrium of a light or heavy string in contact with a smooth or rough plane curve.

(2) Simple Machines.

(3) Graphical methods for the solution of problems in statics.

The more fundamental propositions leading to the development of the subject have been numbered.

Problems in Statics are of very varied kinds but I have made every effort to classify them and put them in different groups in graded form, preceded by typical solved examples. It is hoped that this procedure will greatly assist the student who desires to attempt problems for himself. To encourage

him still further in this attempt, hints for the solution of problems of a more difficult type have been given towards the end of the book.

I hope that in its present form, the book will be more useful not only to the beginner but also to those whose aim is to study the subject for some higher university or competitive examinations.

LAHORE : S. R. GUPTA

March, 1945.

PREFACE TO THE FIRST EDITION

This book has been written in compliance with the demand of certain friends and old students as a companion volume to my book "*Elementary Analytical Dynamics*" published about twelve years ago. It is primarily intended for the use of students preparing for the B. A. Pass and Honours examinations of Indian Universities. It will also prove helpful to students appearing in the various Engineering and Public Service examinations.

This book is an introduction to the subject of Statics and includes such topics as the Principle of Virtual Work, the Stability of Equilibrium and partly the Equilibrium of Strings under Gravity.

An account of the Force of Friction and the laws governing it, has been given in Chapter I on Fundamental Concepts. It has thus been possible to enrich the sets of examples given in later chapters by including problems involving friction.

The subject matter has been comprehensively illustrated by examples. Short sets of examples, for purpose of exercise, are added to the various articles. Some sets of miscellaneous examples and revision questions are also included. All these have been properly graded, some being taken from question papers set at different Indian and Foreign Universities.

Finally, the whole subject has been presented in a simple, precise and systematic manner.

My acknowledgment is generally due to modern writers whose works I have freely consulted. But my particular debt is due to Prof. Shanti Narain of the D. A. V. College for his kindness in going through the manuscript and making valuable suggestions. My thanks are also due to my daughter Miss Sushila Gupta, M. A., lecturer in Mathematics, Fateh Chand College, Lahore for the great help she has rendered in the arrangement of examples and for seeing the book through the press.

LAHORE :
October, 1942

S. R. GUPTA

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CHAPTER 1.

FUNDAMENTAL CONCEPTS.

1.1. A Body is a portion of matter bounded by a closed surface.

A **Force** is an action exerted upon a body which changes or tends to change the body's state of rest or of uniform motion in a straight line.

Mechanics is that branch of science which deals with the action of forces on bodies.

When acted upon by forces, a body may move or remain at rest relative to its surroundings

1.2. Equilibrium. If a system of forces acting upon a body neutralize each other, then the body will either remain at rest or continue to move uniformly (*i.e.*, without acceleration) along a straight line. The forces in such a case are said to be in *equilibrium*. It follows that :—

When a system of forces in equilibrium are added to or subtracted from another system of forces, the effect of the latter remains unchanged.

1.21. The part of Mechanics which deals with a system of forces in equilibrium is called *Statics*.

The part which deals with forces which do not neutralize and therefore cause non uniform motion is called *Dynamics*.

It follows that Statics is only a particular case of Dynamics. But because of its intrinsic interest, it is often studied independently of Dynamics. It also helps to obtain the equations of motion in Dynamics.

1.3. Rigid body. A body is said to be perfectly rigid when the distance between any pair of particles in it remains invariable no matter what forces act on the body.

A perfectly rigid body does not exist in nature. Bodies do change their shape as well as size to some extent under great pressure. A billiard ball, for example, is pressed out of shape during the interval of collision with a second billard ball.

Under ordinary forces, however, these alterations are very slight and in many cases can be completely ignored. Thus in problems where action of forces on bodies is concerned, unless otherwise stated, we assume bodies to be perfectly rigid.

1.31. A **Particle** is a body of infinitely small dimensions.

When we speak of a body as a particle, we mean thereby that for the time being, we are not concerned with its actual dimensions and that we can represent its position simply by a Mathematical point. In certain problems in Astronomy even vast bodies like the earth and the sun are treated as mere particles.

1.4. Measurement of a Force. A force is measured by its effect. In Dynamics we measure a force by the motion it produces in a given body in a given time. If two forces, when applied to the same body for equal intervals of time, generate the same velocity, they are said to be equal. The forces are thus proportional to the velocities they would generate in the same body in equal intervals of time.

In Statics, however, we are not concerned with motion of a body and we conceive of a force as something independent of motion, as an effort of the nature of a *push* or a *pull*. For example, when we hold some object in our hand, some effort on our part is required to support it. This effort or force depends upon the amount of weight supported. The measure of a force in this case, therefore, depends upon the force of gravity, which varies slightly at different places of the earth. In Statics, however, we are concerned merely with the ratio of forces, therefore, we can measure the force by comparing it with some standard weight, say a weight of one pound, or one gramme which is taken as the *unit of force*.

1.41. Representation of a Force. Besides magnitude of a force, we must know its *point of application* and its *direction* (*i.e.*, the direction along a straight line through its point of

application, in which the force tends to move a body to which it is applied.)

Such a quantity which possesses direction as well as magnitude is called a *vector quantity*. It can be represented graphically by a straight line of certain length, drawn in a certain direction, through a given point.

If, for example, the line AB represents the force F, the point A can represent the point of application, the direction of the line represents the direction of the force and length AB of the line can be made to represent the magnitude of the force on some convenient scale.

If instead of AB, we write BA, it represents the same force with its direction reversed.

Two forces represented by BA and AB will balance each other. Thus we have :—

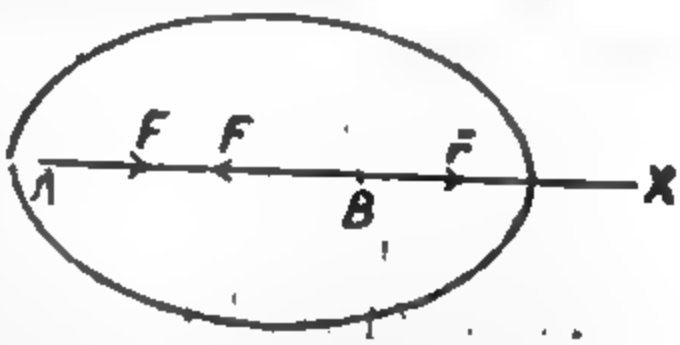
Two forces equal in magnitude, opposite in direction and acting along the same straight line are in equilibrium.

Note 1. The sense of a force represented by straight line, is indicated by the order of the letters used to specify the line.

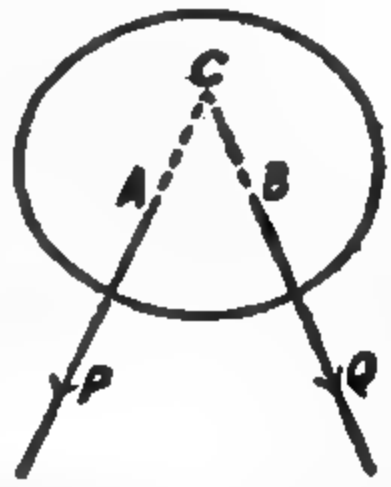
Note 2. The magnitude of a *force* can never be equal to the *length* of a line, but it can be *represented* by a certain length on a certain scale of measurment.

1.42. Transmis ibility of a Force. When a force acts on a body its effect is the same no matter which point in its line of action is taken as the point of application, provided the point is rigidly connected with the body.

Suppose a force F acts at a point A of a rigid body along the line AX. At B, any other point in AX, introduce along the same line, two equal and opposite forces, each equal to F. The force F along AB and the force F along BA, being equal and opposite, cancel each other and we are left with the force F acting at B along BX. Thus we arrive at a force F acting at B along BX instead of the force acting at A along the same line, so that



A force may be supposed to act at any point in its line of action.



From the above it follows that if two forces P and Q act at A and B, two points of a rigid body along the lines which interscet at C, then C can be taken as the point of application of the forces P and Q.

1.43. Resultant and Components. If a single force R can be found which acting alone on a rigid body will have the same effect as a number of forces P, Q, S...etc acting on it, then R is known as the *resultant* of the given forces and the forces P, Q, S...etc. are known as the *components* of R.

1.5. Action and Reaction. *To every reaction there corresponds an equal and opposite reaction.*

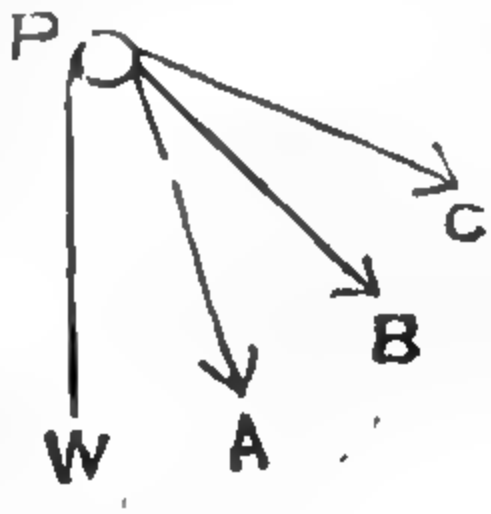
In a system of two bodies, a force never acts on one body without an equal force acting in the opposite direction on the second body. The principle is known as Newton's law of action and reaction.

If a body A exerts a force R on another body B, the body B, exerts an equal force R in the opposite direction on A.

1.6. Tension. When a string is used to support a weight or to drag a body, the force exerted is transmitted to the body through the string. Such a force exerted by means of a string is called *Tension*

If the string is of negligible weight, the *tension throughout the string is the same.* This can be verified by supporting a weight by a string the other end of which is attached to a spring balance. The balance will give the same reading to whatever point of the string it is attached.

Also, if the string supporting the weight W is passed over a smooth peg, the force required at the end of the string to sustain the weight W is the same in whatever direction the string is held so that the tension in the string in all positions (PA, PB, PC, etc.) is the same.



If however, a string be knotted at any of its points to other strings or to weights,

the tension will not, in general, be the same in the different portions separated by the knots.)

161. Tension in an Elastic String. If one end of an elastic string (or a spring) is fixed and a force is applied at the other end, the string extends in length. The relation between the amount of extension and the tension in the string is governed by the law known as **Hooke's Law** which states :—

The tension of an elastic string is proportional to the extension of the string beyond its natural length.

Thus if l be the natural length of the string, x its extension and T, the tension, then

$$T=\lambda \frac{x}{l}.$$

When λ is a constant depending on the material of the string and is known as the **Modulus of Elasticity.**

If the string extends to twice its natural length, the extension x is l and T is consequently equal to λ. *Thus the modulus of elasticity is equal to the amount of tension required to extend the string to twice its natural length.*

Example. *ABC is an elastic string, the unstretched lengths of AB and BC being a and b respectively. A weight W_1 lb. is attached to B and another of W_2 lb is attached to C. The end A is fastened to a fixed point and the string is allowed to hang vertically with the weights at rest. If the modulus of elasticity is λ lb. wt., find the stretched lengths of AB and BC.*

The portion BC of the string has to support only the lower weight and so its tension is W_2.

If x be its stretched length, then by Hooke's law, we have

$$W_2=\lambda \frac{x-b}{b} \quad i.e.\ , x=\frac{bW_2}{\lambda}+b.$$

The portion AB of the string has to support both the weights and, therfore, its tension is W_1+W_2

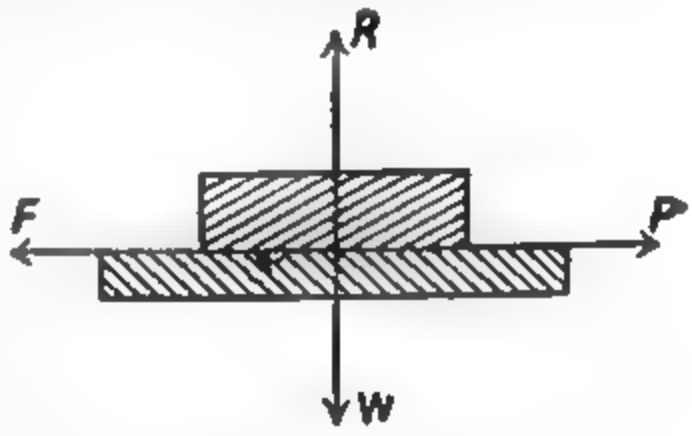
If y be its stretched length, we have

$$W_1+W_2=\lambda \frac{y-a}{a} \quad i.e., y=\frac{a\,(W_1+W_2)}{\lambda}+a.$$

1.7. Attraction is a force which acts upon a body without an apparent medium of transmission. For example, (1) attraction of the earth, (2) mutual attraction of the heavenly bodies which sustains and guides them in their orbits, (3) Attraction of a magnet.

1.8. Friction. Suppose a block of wood is resting on a table. The only forces acting on the block are its weight acting vertically downwards and the reaction of the table. As the block is in equilibrium, the reaction must be equal and opposite to the weight and therefore must act vertically upwards.

Now let a gradually increasing force P be applied to the the block along the table. It will be seen that the block does not at once begin to move and continues to be at rest till P has been sufficiently increased. This shows that there is some force, say F, acting on the block which balances P and thus prevents the block from moving.



The force F is equal and opposite to P and is known as the *force of friction.*

This force is caused by the inequalities in the two surfaces in contact, in virtue of which the two surfaces get interlocked and an effort to cause one to slide on the other causes strain in each.

This force is called into play only when the force P acts. Also it increases with P till the block begins to move. The maximum amount of friction, known as *Limiting friction*, is called into play when the block is just on the point of motion.

The table being rough, the reaction of the table on the block is the resultant of the force of friction F, and the normal pressure R.

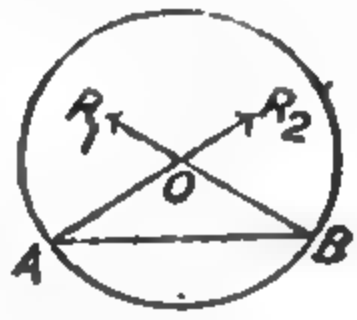
If the table were *smooth*, the block would move, however, small P may be. In this case F does not exist and the reaction of the table is normal to its surface.

1.81. A smooth curve or a smooth surface is one which can exert a pressure only in the direction of the normal.

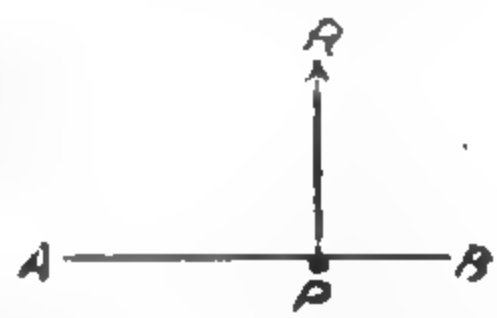
In general *whenever two smooth bodies are in contact, the reaction of either on the other acts along their common normal at the point of contact.*

For example,

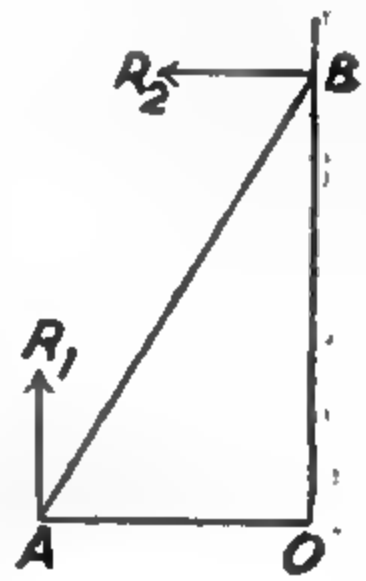
(1) when a rod AB rests inside a smooth sphere, the reactions at the two ends are along the normals at A and B.



(2) Again, if a rod AB rests on a smooth horizontal peg, (or a rail) P, the reaction of the peg on the rod is in the direction perpendicular to the rod at P.



(3) If an end of a rod is in the contact with a smooth wall (or a floor) the reaction of the wall (or the floor) is perpendicular to the wall (or the floor) at the point of contact.



1.9. Law of friction. Experiments such as given in Art 1.8 lead to the formulation of certain laws, which are given below.

1.91. Statical Friction. The friction acting on a body, which is at rest is called *statical friction* and is governed by the following laws :—

(1) The direction of friction is opposite the direction in which the body tends to move.

(2) The magnitude of the force of friction is exactly equal to the force tending to produce motion.

1.92. Limiting Friction. The friction acting on the body

when it is just on the point of motion is called *Limiting friction.* It obeys the following laws :—

(1) Limiting friction is equal in magnitude and opposite in direction to the force which tends to produce motion.

(2) The ratio of the amount of limiting friction to the normal pressure is constant.

This constant is known as the **coefficient of friction** and is usually denoted by the letter μ.

If F be the amount of limiting friction and R, the normal pressure then $F/R=\mu$ or $F=\mu R$.

(3) The coefficient of friction depends entirely on the material of the bodies in contact and is independent of the area and shape of the surfaces in contact.

1.93. Dynamical Friction is the friction which acts on the body when it is actually in motion. It acts in the direction opposite to the direction of motion of the body. Its ratio to the normal pressure is slightly *less than* the confficient of friction.

1.94. The force of friction between a body and a rough surface belongs to a class of force known as *passive resistance i.e.* forces which come into play on account of the action of other forces and which always tend to destroy the effect of these other forces.

In nature, the force of friction plays an important part and cannot be dispensed with. It is in the presence of this force between the feet and the ground that renders walking possible. If the surface be a smooth one, it is very difficult to walk on it. Again, it is the force of friction that propels the driving wheels of an engine ; if there were no friction between the wheels and the rails, the driving wheele would slip round and rotate, the engine remaining stationary. [See author's Dynamics Art. 3.6 (6)].

EXAMPLES I.

1. A man of weight 160 lb. holds one end of a rope which passes over a pulley vertically above his head and to the other end of the rope is attached a weight of 120 lb. Find the force with which the man's feet press on the floor. *Ans.* 40 lb.

2. In a tug of war there are 10 members pulling on each side. If the average force exerted by each member on either side is 110 lb. wt., find the tension in the rope.

(*Ans.* 1100 lb. wt.)

3. A mass of 10 lb. rest on a rough table whose co-efficient of friction is $\frac{1}{2}$. A string attached to it and passing over the edge of the table, carries another weight at the other end. Find the magnitude of this mass if the mass on the table is on the point of motion.

(*Ans.* 5 lb.)

4. A wooden plank 5 ft. by 2 ft. of weight 100 lb. is lying on the ground. Find the pressure per unit area offered by the ground.

If the plank is on the point of motion when a horizontal force of 75 lb. wt. acts on it. Find the coefficient of friction.

(*Ans.* 10 lb. ; $\frac{3}{4}$.)

5. An elastic cord whose natural length is 8 inches, can be kept stretched to a length of 12 inches by a force of 5 lb. wt. Show that the modulus of elasticity of the cord is 10 lb. wt.

6. ABC is an elastic string which when unstreched is such that AB=8 in. and BC=3 in. A particle weighing 4 lb. is attached to B and another of 2 lb. is attached to C. The end A is fastened to a fixed point and the string is allowed to hang vertically with the particles at rest. If the modulus of elasticity is 6 lb. wt., find the stretched lengths of AB and BC.

(*Ans.* 16 in., 4 in.)

7. An elastic string ABC, whose modulus of elasticity is 4 W, is tied at A to a fixed point. At B and C are attached weights each equal to W, the unstreched lengths of AB and BC being each equal to *c*. Show that, if the string and bodies take up a vertical position of equilibrium, the streched lengths of AB and BC are $\frac{3c}{2}$ and $\frac{5c}{4}$ (P. U. 1938.)

8. A uniform elastic string has a length a_1 when the tension is T_1 and a length a_2 when the tension is T_2. Show that its natural length is $\frac{(a_2T_1-a_1T_2)}{T_1-T_2}$ (P. U. 1944.)

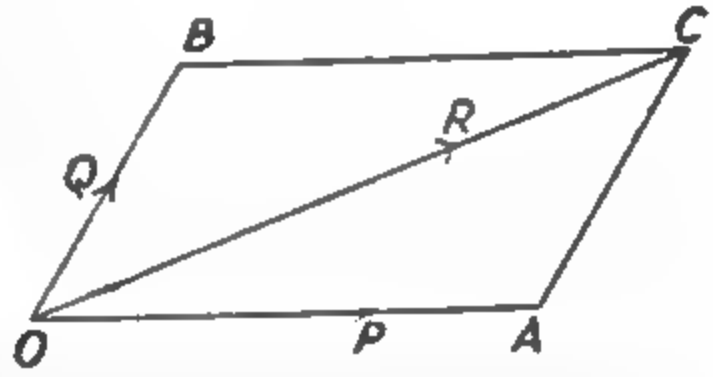
CHAPTER II

FORCES ACTING AT A POINT I.

2. 1. When we speak of forces acting at a point, it is understood that they are acting on a particle placed at that point.

2. 11. Theorem I. Resultant of two forces. *If two forces acting at a point are represented, in magnitude and direction, by the sides of a parallelogram drawn from the point, their resultant is represented, both in magnitude and direction, by the diagonal of the parallelogram drawn through that point.*

Thus if two forces P and Q acting at the point O are represented in magnitude and direction, by the lines OA and OB, their resultant R is then represented by the diagonal OC.



The solution of the various problems in Statics is based on this fundamental principle, known as the *law of the parallelogram of forces.*

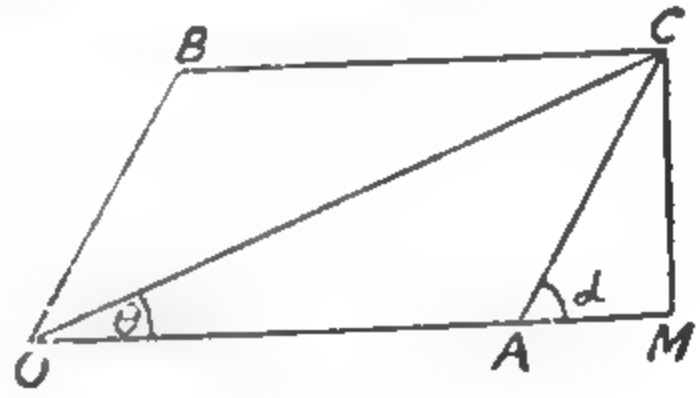
The best proof of this is based on the laws of Dynamics. Here its truth is assumed on experimental basis

2 12. Magnitude and direction of the Resultant.

If $\angle AOB = \alpha$, then from the $\triangle OAC$, we have

$$OC^2 = OA^2 + AC^2 - 2OA.AC \cos \angle OAC.$$
$$= OA^2 + OB^2 - 2OA.OB \cos (180^\circ - \alpha).$$
$$= OA^2 + OB^2 + 2OA.OB \cos \alpha.$$

i.e $R^2 = P^2 + Q^2 + 2PQ \cos \alpha$ ...(1)



Again if $\angle AOC = \theta$, and CM is $\perp$ to OA, we have

$$\tan \theta = \frac{MC}{OA + AM} = \frac{AC \sin \alpha}{OA + AC \cos \alpha} = \frac{Q \sin \alpha}{P + Q \cos \alpha} \quad \ldots(2).$$

and $\sin\theta=\frac{MC}{OC}=\frac{Q\sin\alpha}{R}$; $\cos\theta=\frac{OM}{OC}=\frac{P+Q\cos\alpha}{R}$.

Equations (1) and (2) give the magnitude and direction of the resultant.

In the particular case when $P=Q$, we have

$R^2=2P^2+2P^2\cos\alpha$, *i.e.*, $R=2P\cos\frac{\alpha}{2}$.

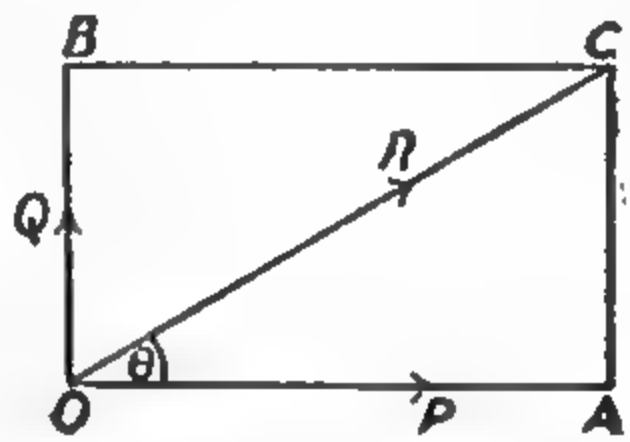
and $\tan\theta=\frac{P\sin\alpha}{P+P\cos\alpha}=\tan\frac{\alpha}{2}$.

i.e., *the line of action of the resultant force bisects the angle between its equal components.*

Again in another case when P and Q are at right angle, *i.e.*, $\alpha=90°$, we get

$$R=\sqrt{P^2+Q^2} \text{ and } \tan\theta=\frac{Q}{P}$$

which is also evident directly from the figure.



2.13. Example (i) *The resultant of forces 2P and 3P acting at a point is $\sqrt{7}P$, find the angle between the forces and the direction of the resultant.*

By (1), we have

$$(\sqrt{7}P)^2=(3P)^2+(2P)^2+2.3P.2P\cos\alpha,$$

so that $\cos\alpha=-\frac{1}{2}$, *i.e.*, $\alpha=120°$

and by (2) $\tan\theta=\frac{2P\sin\alpha}{3P+2P\cos\alpha}=\frac{2P.\frac{\sqrt{3}}{2}}{3P-2P.\frac{1}{2}}=\frac{\sqrt{3}}{2}$

Example (ii) *The resultant of two forces P, Q acting at a certain angle is F, and that of P, R acting at the same angle is also F, the resultant of Q, R, again acting at the same angle is G. Prove that if $P+Q+R=O$, $G=F$.* *(P. U. 1938.)*

Let the given angle be α, then, by the question.

$F^2=P^2+Q^2+2PQ\cos\alpha$ ...(1)

$F^2=P^2+R^2+2PR\cos\alpha$ ...(2)

and $G^2=Q^2+R^2+2QR\cos\alpha$ ...(3)

From (1) and (2), $Q^2-R^2+2P(Q-R)\cos\alpha=0$

Since $Q-R\neq0$ and $P+Q+R=0$ *i.e.*, $Q+R=-P$, we get

$$\cos\alpha=-\frac{Q+R}{2P}=\frac{P}{2P}=\frac{1}{2},\ i.e.,\ \alpha=60°.$$

Now from (2) and (3).

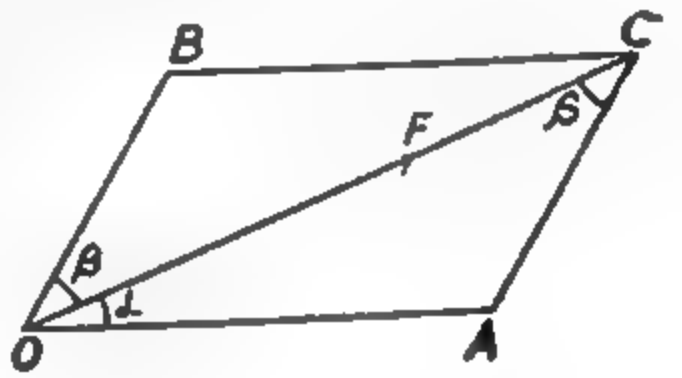
$$F^2-G^2=P^2-Q^2+2R(P-Q)\cos 60°=(P-Q)(P+Q+R)=0.$$

Hence $$F=G.$$

2.14. Components of a force in two given directions. A force may be resolved into two component forces in an infinite number of ways, for, an infinite number of parallellograms can be constructed having the straight line representing the force as the diagonal.

The components of the force F in directions, making angles α, β with its line of action on opposite sides of it can be found as follows :

Let OC represent the force F. Draw OA, OB making angles α, β with OC and complete the parallelogram OACB, then OA and OB represent the required components of F.



From the $\triangle$OAC, we have

$$\frac{OA}{\sin\beta}=\frac{AC}{\sin\alpha}=\frac{OC}{\sin\angle OAC}=\frac{F}{\sin(\alpha+\beta)}$$

$$\text{Hence } OA=\frac{F\sin\beta}{\sin(\alpha+\beta)} \text{ and } OB=AC=\frac{F\sin\alpha}{\sin(\alpha+\beta)}.$$

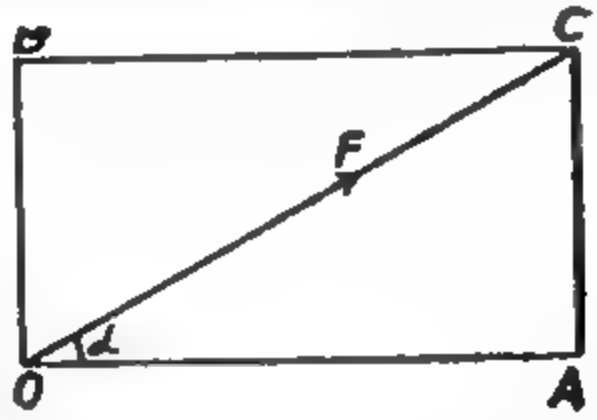
Example. *Resolve a force F along two lines making angles 45° and 15° with its direction.*

Here $\alpha=15°$, $\beta=15°$, $\alpha+\beta=60°$.

$\therefore$ components of the force are,

$$\frac{F\sin 45°}{\sin 60°}=\frac{2F}{\sqrt{6}} \text{ and } \frac{F\sin 15°}{\sin 60°}=\frac{(\sqrt{3}-1)F}{\sqrt{6}}.$$

2.15. In the particular case when the components are at right angles, *i.e.*, $\alpha+\beta=90°$ or $\beta=90°-\alpha$, we have



$OA=F\cos\alpha$;

$OB=F\sin\alpha=F\cos(90°-\alpha)$

which is otherwise evident from the figure.

In this case, the components OA and OB are called the *resolved parts* of F.

It follows that the *resolved part of a force in a certain direction is equal to the product of the force and the cosine of the angle between its line of action and the given direction.*

As $\cos 90°=0$, a force has no resolved part in a direction perpendicular to its line of action.

The component of the force F in a given direction OA is not the same as its resolved part. To find the components of a force we must be told both the directions in which the force is to be decomposed, but to find the resolved parts, we need know only one direction, as the second direction must be at right angles to the known direction.

EXAMPLES II

1. The resultant of two forces P and 2P, acting at a point is perpendicular to P. Find the angle between P and 2P.

[*Ans.* 120°]

2. The resulant of two forces P and Q is equal to Q in magnitude and that of P and 2Q (acting in the same direction as before) is also Q. Find the magnitude of P and the angle between the forces P and Q. *Ans.* $\sqrt{3}$ Q, 150°.

3. The resultant of two forces P_1 and P_2 acting at right angles is R. If P_1, P_2 be each increased by 3 lb, R is increased by 4 lb and is now equal to the sum of the original values of P_1, and P_2. Find P_1 and P_2. [*Ans.* 5 lb., 12 lb.]

4. Two forces one of which is double the other, have a certain resultant such that if the greater force be doubled again and the smaller be increased by 4 lb. wt, the resultant has the same direction as before. Find the forces and also find how the resultant is altered. [*Ans.* 4., 8 ; Resultant doubled.].

5. When two equal forces are inclined at an angle 2θ, their resultant is twice as great as when they are inclined at an angle 2ϕ, show that

$$\cos\theta = 2\cos\phi.$$

6. The resultant of P and Q is R; if Q is doubled, R is doubled; if Q be reversed, R is again doubled. Show that

$$P^2 : Q^2 : R^2 = 2 : 3 : 2 \qquad \text{(U. U.)}$$

7. The resultant of two forces P and Q (P > Q) divides the angle between their directions into two parts in the ratio 1 : 2. Find the angle between the forces and show that the magnitude of the resultant is $(P^2 - Q^2)/Q$.

[*Ans.* $3\cos^{-1}(P/2Q)$].

8. Two forces P+Q, P−Q makes an angle 2α with one another, and their resultant makes an angle θ with the bisector of the angle between them. Show that $P\tan\theta = Q\tan\alpha$.

(P. U., 1931.)

9. If forces P and Q acting at an angle θ be interchanged in position, show that the resultant turns through an angle ϕ such that

$$\tan\frac{\phi}{2} = (P-Q)\tan\frac{\theta}{2} \;/\; (P+Q).$$

10. If the greatest possible resultant of two forces P and Q is m times the least, show that the angle θ between them when their resultant is half their sum is given by

$$\cos\theta = -\frac{m^2+2}{2\,(m^2-1)}.$$

11. Three forces P, Q, R acting at a point are in equilibrium; one of them P is turned round this point through an angle A, find the direction of the resultant of the three forces.

If the inclination of P continue to alter, show that the inclination of the resultant alters by half the amount.

[*Ans.* It makes $90° + \frac{1}{2}$ A with the original direction of P]

12. A force P acting at a point O is given in magnitude and direction and is represented by a vector drawn from O. If a second force Q acting at O is given in magnitude only, find the

locus of the extremity of the vector drawn from O which represents the resultant of P and Q.

Show that if P is greater than Q, the inclination of the resultant to the direction of P cannot exceed θ where

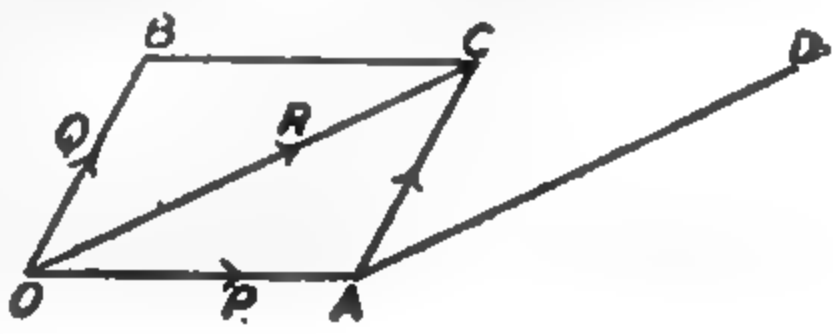
$$\sin\theta = \frac{Q}{P}.$$

13. The resolved part of the resultant R of two forces P and Q in the direction of P is of magnitude Q. Show that the angle between the forces is $2\sin^{-1}\sqrt{\frac{P}{2Q}}$.

14. The greatest resultant that two forces can have is of magnitude P, and the least is of magnitude Q. Show that when they act at angle α, their resultant is of magnitude

$$\sqrt{P^2\cos^2\frac{\alpha}{2} + Q^2\sin^2\frac{\alpha}{2}}.$$

2.2. In art. 2.11, since AC, is equal and parallel to OB, therefore, so far as magnitude and direction are concerned, the force Q can be represented by the line AC, so that OC represents the resultant of forces represented by OA and AC.



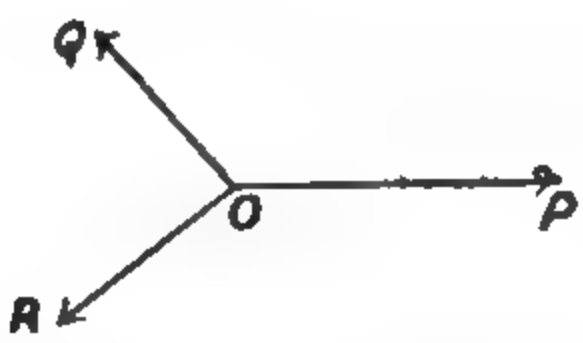
Hence if two forces acting at a point are represented by the two sides OA and AC, drawn in the same sense, of the $\triangle$OAC, their resultant is represented, in magnitude and direction by the third side taken in the opposite sense (*i.e.*, OC).

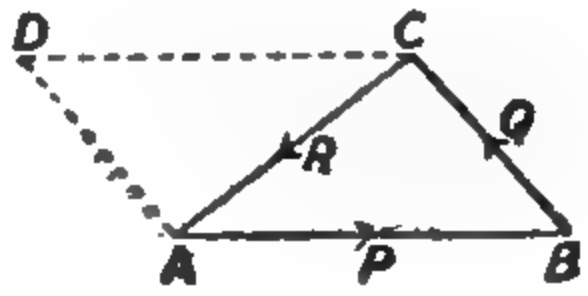
If, however, the forces represented by OA and AC, are actually acting along these lines, the resultant though represented by OC, in magnitude and direction, actually acts along, AD, the line through A parallel toOC.

2.21. Triangle of Forces. *If the forces acting at a point are represented, in magnitude and direction, by the three sides of a triangle, taken in order, they are in equilibrium.*

Let the three forces, P,Q,R, act at the point O, and be represented in magnitude, and direction, by the sides AB, BC and

CA of the $\triangle ABC$. Complete the parallelogram ABCD. AD being parallel and equal to BC represents the force Q and therefore,





by parallelogram of forces, the resultant of P and Q is represented by AC, therefore the resultant of P, Q and R is the same as the resultant of forces represented by AC and CA, which being equal and opposite, cancel and hence P, Q, R are in equilibrium.

2 22. Converse of the triangle of Forces. If three, forces, acting at a point are in equilibrium, they can be represented by the sides of a triangle taken in order, the triangle being drawn so as to have its sides respectively parallel to the direction of the forces.

Let the forces P, Q, R acting at the point O be in equilibrium. Draw AB parallel to the direction of force P, and on some scale, let AB represent the force P. Again draw BC parallel to the direction of force Q, and on the same scale, let BC represent Q; join CA.

By Art. 2 2, the resultant of P and Q is represented by AC. But since P, Q, R are in equilibrium, R must be equal and opposite to the resultant of P and Q. Therefore R must be represented by CA.

Hence P, Q, R are, respectively, represented by the sides AB, BC and CA of the $\triangle ABC$.

Cor. *If three forces acting at a point are in equilibrium, they must be coplanar.*

2.23. If three forces proportional to the sides of a triangle act at a point in directions perpendicular to the sides taken in order, they shall be in equilibrium.

For, if the triangle be turned through a right angle its sides taken in order will be parallel to the respective forces and

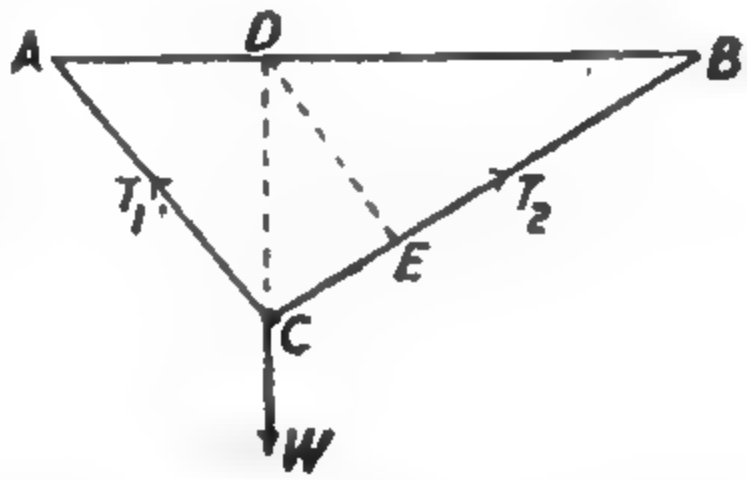
will, therefore, represent the forces in magnitude and direction. Hence by Art. 2.21 the forces are in equilibrium.

This is a particular case of the following general theorem :-

If three forces acting at a point are proportional to the sides of a triangle and the directions of the forces make the same constant angle, measured in the same sense, with the sides respectively, the forces are in equilibrium.

Example (i). *A weight W is suspended by two strings at right angles to one another and attached to points in the same horizontal line; prove that their tensions are inversely proportional to their lengths.*

The strings CA, CB are attached to points A and B in the same horizontal line such that $\angle ACB = 90°$.



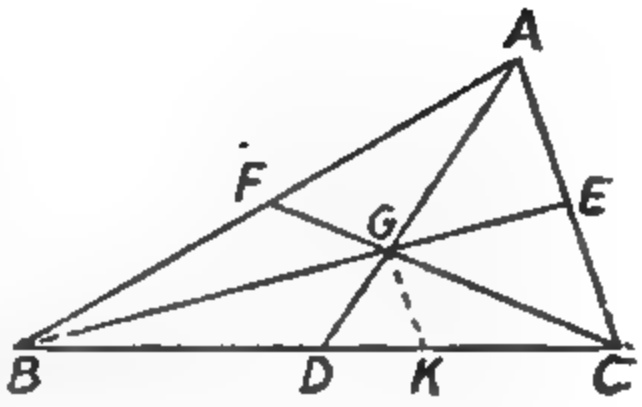
The weight W and the tensions T_1 and T_2 are respectively perpendicular to the sides AB, BC, CA of the $\triangle ABC$, which when turned through a right angle represents the triangle of forces. The forces are therefore proportional respectively to the sides perpendicular to them.

$$\text{Hence } \frac{T_1}{T_2} = \frac{BC}{AC} = \frac{1}{AC} \div \frac{1}{BC}$$

Aliter Let the line of action of W meet AB in D. Draw DE parallel to AC. CDE is the triangle of forces
$\therefore T_1 : T_2 = DE : CE = BC : AC$ ($\triangle$S ABC, CDE being similar.)

Example. (ii) *ABC is a triangle and D, E, F are the middle pionts of the sides. Forces represented by AD, $\frac{2}{3}$ BE and $\frac{1}{3}$ CF act on a particle at the point where AD and BE meet. Show that the resultant is represented in magnitude and direction by $\frac{1}{2}$ AC, and that its line of action divides BC in the ratio 2 : 1.*
(P. U., 1933.)

AD and BE meet at G, the centroid of the $\triangle$ABC.



Force $\frac{2}{3}$BE is equivalent to BG, and $\frac{1}{3}$CF to GF.

From $\triangle$BGF, forces BG and GF $\equiv$BF$\equiv\frac{1}{2}$BA.

Again force AD

$\equiv$AC and CD or AC and $\frac{1}{2}$CB.

$\therefore$ the given forces are

$\equiv\frac{1}{2}$BA, AC, $\frac{1}{2}$CB

$\equiv\frac{1}{2}$BA, $\frac{1}{2}$AC, $\frac{1}{2}$CB and $\frac{1}{2}$AC.

$\equiv\frac{1}{2}$AC.

For, forces $\frac{1}{2}$BA, $\frac{1}{2}$AC, $\frac{1}{2}$CB being proportional to the sides of the $\triangle$ABC and acting at the point G are in equilibrium.

Hence the resultant is $\frac{1}{2}$ AC in magnitude, and direction.

Since the resultant is to act at G, it acts along GK drawn parallel to AC, meeting BC in K.

From $\triangle$s BGK and BEC, we get

$$\frac{BK}{KC}=\frac{BG}{GE}=\frac{2}{1},$$

so that K divides BC in the ratio 2 : 1.

EXAMPLES III.

1. ABCD is a parallelogram and P is any point, not necessarily in its plane. Prove that the system of forces represented by AP, PB, CP and PD is in equilibrium.

2. ABCD is a parallelogram. Prove that the forces completely represented by the lines AB, CD, AC and DB have a resultant represented by 2 AB, in magnitude, direction and position.

3. The resultant of two forces P and Q acting at a certain angle is R, the resultant of P and Q′ acting at the same angle is R′; show that the resultant of R and a force equal and opposite to R′ is the difference of Q and Q′.

4. The ends of a string 17 inches long are attached to two points 13 inches apart in the same horizontal line and a weight of 26 lb is attached to the string 5 inches from one end. Find the tension in each portion of the string. [*Ans.* 24, 10 lb.]

5. ABC. is a triangle and P is any point in BC. If PQ represents the resultant of forces represented by AP, PB and BC, show that the locus of Q is a straight line parallel to BC.

If the forces are represented by AP, PB, and PC, show that Q is a fixed point on the median through A.

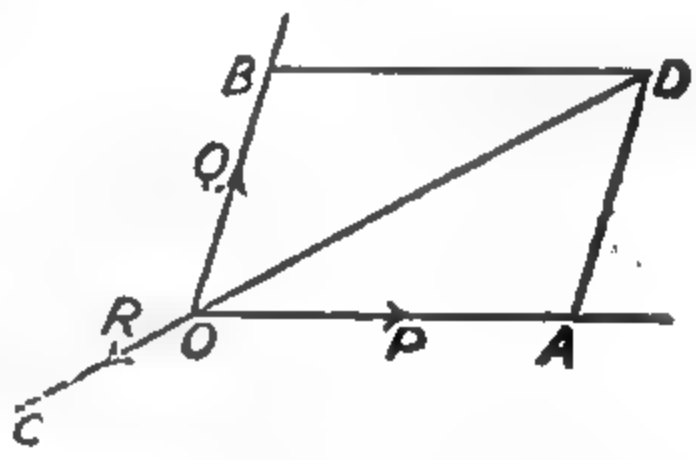
6. Three coplanar forces P, Q, R act at the vertices A, B, C respectively of a $\triangle ABC$, in which AB : BC : CA = 4 : 3 : 2. Q and R are parallel to CA and BA and the magnitude of R is 10 lb. weight. Find the magnitude of P and Q in order that the forces may be in equilibrium.

[*Ans.* $\frac{5}{2}\sqrt{31}$ lb. wt., 5 lb. wt.

7. If ABCDEF is a regular hexagon and O any point in its plane, show that forces represented, in magnitude and direction, by OA, OC, OE, BO, DO and FO are in equilibrium.

2.3. Lami's Theorem. *If three forces acting at a point are in equilibrium, each is proportional to the sine of the angle between the other two.*

Let the forces P, Q, R acting at O, be in equilibrium, and let P and Q be represented, on some scale. by OA and OB respectively. Complete the parallelogram OADB.



Now since P, Q, R are in equilibrium. therefore OD which represents the resultant of P and Q, must be equal and opposite to R, *i.e.*, the force R is represented by DO. Also AD being equal and parallel to OB represents Q.

Hence P, Q, R are represented by the sides OA, AD and DO of the triangle OAD.

$$\therefore \quad \frac{P}{OA} = \frac{Q}{AD} = \frac{R}{OD},$$

The sides of a triangle being proportional to the sines of the opposite angles, we have

$$\frac{P}{\sin \angle ODA} = \frac{Q}{\sin \angle AOD} = \frac{R}{\sin \angle OAD}$$

But $\angle ODA = \angle BOD = \pi - \angle QOR$. $\angle AOD = \pi - \angle POR$,

and $\angle OAD = \pi - \angle POQ$,

Hence $$\frac{P}{\sin \angle QOR} = \frac{Q}{\sin \angle POR} = \frac{R}{\sin \angle POQ} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

2.31. Converse of Lami's Theorem. *If three coplanar forces acting at a point are such that each is proportional to the sine of the angle between the directions of the other two, the forces are in equilibrium.*

Let P, Q, R be the given forces satisfying result (1), we have to prove that they are in equilibrium.

Let P′, Q′, R′ be a set of forces in *equilibrium* acting along the same lines, then by Lami's Theorem, we have

$$\frac{P'}{\sin \angle QOR} = \frac{Q'}{\sin \angle POR} = \frac{R'}{\sin \angle POQ} \quad \ldots\ldots(2)$$

∴ from (2) and the given result (1), we get

$$\frac{P}{P'} = \frac{Q}{Q'} = \frac{R}{R'}.$$

Hence the sides of the triangle representing the forces P′, Q′, R′ will also represent the forces P. Q. R (on a different scale) which are, therefore, in equilibrium.

Cor. *If one set of concurrent forces (P, Q, R) acting in three given directions is in equilibrium all other sets acting in equilibrium in the same directions are merely multiples of the set (P, Q, R).*

* **2.32.** Lami's Theorem can also be proved by resolving forces in directions perpendicular to two of the lines of action of the forces.

The forces being in equilibrium, the sum of the resolved parts along any line in their plane must vanish.

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* This may be read after Art 2.4 has been studied.

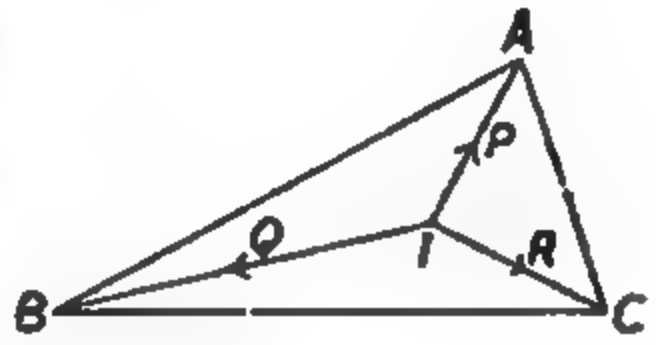
Resolving along a line perpendicular to OA, [Seo Fig. Art. 2.3] we have

$$Q \sin \angle POQ - R \sin \angle POR = 0 \text{ i.e., } \frac{Q}{\sin \angle POQ} = \frac{R}{\sin \angle POR}$$

A similar result is obtained by resolving along the line perpendicular to OB or OC-

Example : (i) *Forces P, Q, R acting along IA, IB, IC where I is the incentre of* $\triangle ABC$, *are in equilibrium, show that*

$$P : Q : R = Cos\frac{A}{2} : Cos\frac{B}{2} : Cos\frac{C}{2}$$



By Lami's Theorem, we have

$$\frac{P}{\text{Sin } \angle BIC} = \frac{Q}{\text{Sin } \angle AIC} = \frac{R}{\text{Sin } \angle AIB}$$

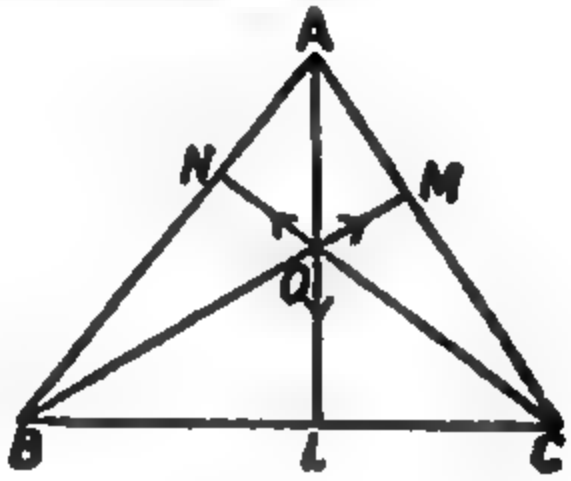
But $\left(\angle BIC = 180° - \frac{B+C}{2} = 90° + \frac{A}{2} \therefore \text{ Sin } \angle BIC = \text{Cos } \frac{A}{2} \right.$.

Similarly $\sin \angle AIC = \text{Cos } \frac{B}{2}$ and $\text{Sin} \angle AIB = \text{Cos } \frac{C}{2}$

Hence $$\frac{P}{\text{Cos } \frac{A}{2}} = \frac{Q}{\text{Cos } \frac{B}{2}} = \frac{R}{\text{Cos } \frac{C}{2}} \left. \right)$$

Example (ii) *Three forces proportional to the sides of a triangle act at the verbices towards and perpendicular to the corresponding sides, prove that the forces are in equilibrium.*

The forces P, Q, R respectively acting along AL, BM, CN, meet at O, the orthocentre of the $\triangle ABC$; then since



$$\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$$

$$\therefore \quad \frac{P}{\text{Sin A}} = \frac{Q}{\text{Sin B}} = \frac{R}{\text{Sin C}}$$

But $A = \pi - \angle MON$, $B = \pi - \angle NOL$, $C = \pi - \angle LOM$

$$\therefore \quad \frac{P}{\text{Sin } \angle MON} = \frac{Q}{\text{Sin } \angle NOL} = \frac{R}{\text{sin } \angle LOM}$$

Hence by the Converse of Lami's Theorem, the forces P, Q, R are in equilibrium.

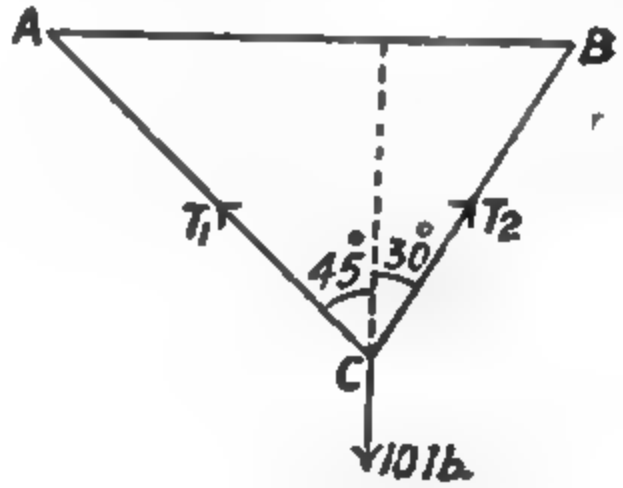
Note. The result is otherwise obvious, for, the forces meet at a point and can be represented by the sides of the triangle ABC turned through a right angle, and hence they satisfy the necessary conditions of equilibrium.

Example (iii) *Three forces act at the middle points of the sides of a triangle, all inwards or all outwards. If each force is perpendicular and proportional to the side at which it acts, show that the forces are in equilibrium.*

For, they satisfy the condition of equilibrium of three forces, viz (*a*) they meet at a point (the circumcentre) and (*b*) they satisfy Lami's Theorm.

Example (iv) *A body weighing 10 lb. is suspended from a horizontal beam by two strings. If the inclinations of the strings to the vertical are 45° and 30°. Find the tension in each string.*

AB is the beam, the weight of 10 lb. is attached at C to the strings CA and CB as shown. The point C is in equilibrium under three forces T_1, T_2, and 10 lb. weight.



Hence by Lami's Theorem we have

$$\frac{T_1}{\text{Sin } 30°} = \frac{T_2}{\text{Sin } 45°} = \frac{10}{\text{Sin } 75°}$$

i.e. $$T_1 = \frac{10 \text{ Sin } 30°}{\text{Sin } 75°} = 5(\sqrt{6}-\sqrt{2}) = 5{\cdot}18 \text{ lb.}$$

$$T_2 = \frac{10 \text{ Sin } 45°}{\text{Sin } 75°} = 10\ (\sqrt{3}-1) = 7{\cdot}32 \text{ lb.}$$

EXAMPLES IV

1. A 5 lb. weight hangs freely at the end of a string. Find what horizontal force applied to the weight will keep the string inclined at 30° to the vertical. [*Ans.* 2·9 lb. wt. nearly.

2. A weight of 24 lb. is suspended by two flexible strings

one of which is horizontal and the other inclined at an angle of 30° to the vertical, what is the tension in each string ?

[*Ans.* $8\sqrt{3}$; $16\sqrt{3}$ lb. wt.

3. Three forces P, Q, R acting at a point are in equilibrium and the angle between P and Q is double of the angle between P and R. Prove that $R^2=Q\ (Q-P)$.

4. A weight is supported by two strings fastened to two points in the same horizontal line, the strings being equally strong, but one longer than the other. If more weights be continually added to the first one, which string will break first ?

[*Ans.* The shorter.]

5. Find the greatest weight which can be supported by two light strings making angles 60° and 45° with the vertical, it being known that either string will break under a tension of W lb. wt (P. U)

(*Ans.* $\frac{W}{\sqrt{6}}\ (\sqrt{3}+1)$

6. A weight W is tied at A to two strings AB, AC, B and C being in the same level, and hangs freely. If AB : AC : BC= 2 : 3 : 4, find the tensions in the strings in terms of W.

[*Ans.* $\frac{7}{2\sqrt{15}}$ W, $\frac{11}{4\sqrt{15}}$ W.]

7. The ends of an elastic string of modulus $5\sqrt{3}$ lb., are attached to two points A and B in a horizontal line and initially the string is just taut and unstreched. A particle of weight 15 lb. is attached to the middle point of the string. Prove that in the position of equilibrium each portion of the string makes 30° with the vertical.

8. A bead of weight W can slide on a smooth circular wire in a vertical plane. The bead is attached by a light inextensible string to the highest point of the wire and in equilibrium tne string is taut and makes an angle θ with the vertical. Find T, the tension of the string and R the reaction of the wire on the bead.

[*Ans.* T=2W Cos θ, R=W.]

9. Forces P, Q, R acting along OA, OB, and OC, O being the circumcentre of the $\triangle$ABC are in equilibrium prove that

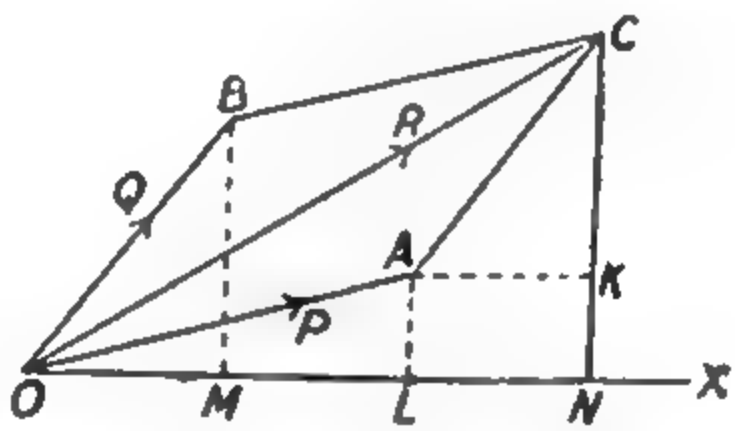
$$P : Q : R = a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2)$$

10. A, B, C, are three points on the circumference of a circle and forces P and Q act along AB and BC respectively. If the resultant force R acts along the tangent at B, show that

$$\frac{P}{BC}=\frac{Q}{AB}=\frac{R}{AC}.$$

2 4. Theorem II A. *The sum of the resolved parts of two concurrent forces along a given line in their plane is equal to the resolved part of their resultant along the same line.*

Let OA and OB represent two forces P and Q. Complete the parallelogram OACB, then the diagonal OC represents the resultant force R.



Let OX be the given line in the plane AOB, draw AL, BM, CN perp. to OX and AK perp. to NC.

Then the sum of the resolved parts of P and Q along OX=OL+OM=OL+AK=OL+LN

=ON.=the resolved part of R along OX.

By its repeated application, the theorem can be easily extended to the more general theorem which may be stated as :—

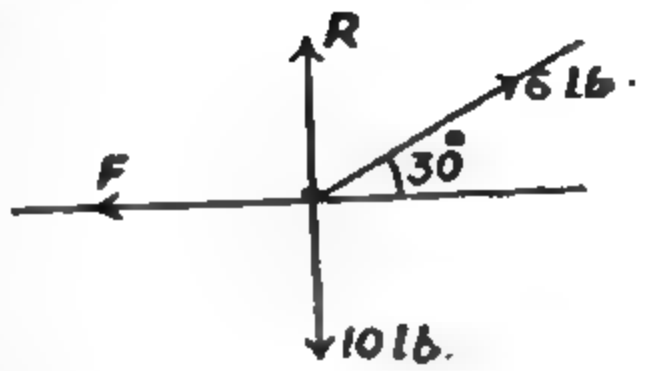
The sum of the resolved parts of a number of concurrent and coplanar forces along a given line (in their plane) is equal to the resolved part of their resultant along the same line.

Cor. *If the forces are in equilibrium, the sum of the resolved parts vanishes.*

EXAMPLES V

1. *A body of weight 10 lb. rests on a rough horizontal table and a force of 6 lb. acts on it upwards in a direction making an angle of 30° with the table. Find the normal reaction of the table and the amount of friction on the body.*

Let R be the normal reaction on the body (regarded as a particle) and F, the force of friction.



The body being at rest, the sum of the resolved parts of the forces acting on it along any line in the plane of the forces must vanish.

∴ Resolving horizontally and vertically, we have

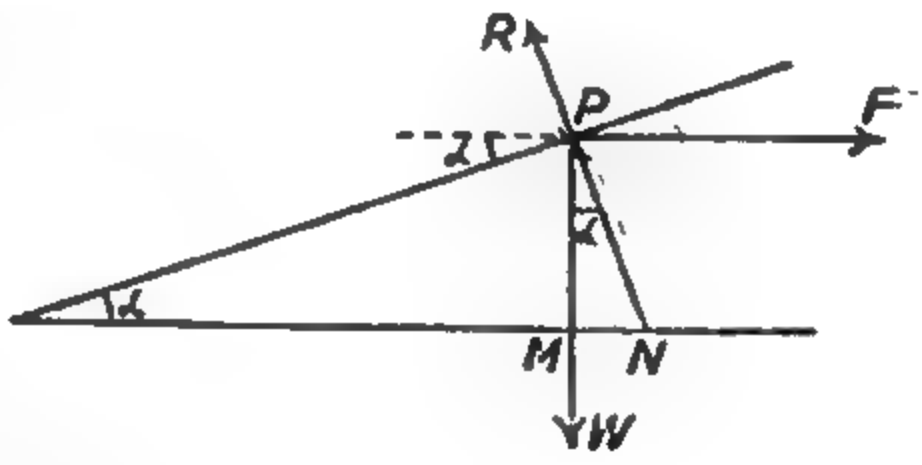
$$6\cos 30° - F = 0 \qquad i.e.,\ F = 3\sqrt{3} \text{ lb.}$$

and $$R + 6\sin 30° - 10 = 0 \qquad i.e.,\ R = 10 - 3 = 7 \text{ lb.}$$

Note.—If the force of 6 lb. does not act, no force of friction will be called into play and the pressure will be normal and equal to the weight of the body. Thus the horizontal component of the force balances the self-adjusting friction and the vertical component decreases the normal pressure.

2. *A body of weight W rests on a smooth inclined plane of inclination α. Find the horizontal force required to keep it in equilibrium and the reaction of the plane.*

Let F be the required force and R the reaction of the plane.



Resolving vertically and horizontally, we have

$$R\cos\alpha - W = 0$$

i.e., $$R = W\sec\alpha$$

and $$F - R\sin\alpha = 0$$

i.e., $$F = R\sin\alpha = W\tan\alpha.$$

We can obtain the same result from the principle of the triangle of forces.

The triangle PMN whose sides PM, MN, NP are respectively parallel to the forces W, F and R, is the triangle of forces, so that we have

$$\frac{W}{PM} = \frac{F}{MN} = \frac{R}{NP}$$

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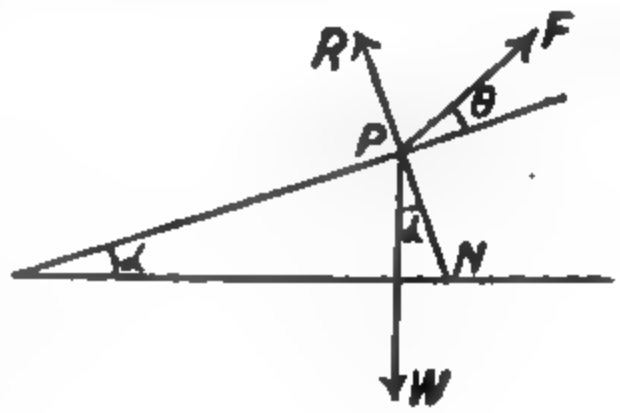
$$\therefore \qquad R = W.\frac{NP}{PM} = W \sec \alpha$$

and $$F = W.\frac{MN}{PM} = W \tan \alpha.$$

Note.—In all such problems it will be understood that the force F acts in a vertical plane through the line of the greatest slope through P.

3. *The force F (Ex. 2) acts in a vertical plane in a direction making an angle θ with the plane (i, e., with the line of the greatest slope through P). Find the relation between F, R and W.*

Resolving along and perpendicular to the plane, we have



$$F \cos\theta - W \sin\alpha = 0$$

i.e., $$F = W \sin\alpha \sec\theta$$

and $$R + F \sin\theta \quad W \cos\alpha = 0$$

i.e., $$R = W\cos\alpha - F\sin\theta$$

$$= W\cos\alpha - \frac{W \sin\alpha \sin\theta}{\cos\theta}$$

$$= \frac{W\cos(\alpha+\theta)}{\cos\theta}$$

Cor. (*a*) *In particular when $\theta = 0$ i. e., F acts along the plane, we get $F = W \sin\alpha$ and $R = W \cos\alpha$*

This is otherwise obviovs, W sin α, W cos α being the Components of W along and perp. to the plane.

(*b*) *Again when F acts horizontally $\theta = -\alpha$, so that $F = W \tan\alpha$ and $R = W \sec\alpha$, the same as o tained in Ex. 2.*

4. A body of mass 20 lb. is at rest on a rough horizontal plane when a force of $6\sqrt{3}$ lb. is acting on it in an upward direction making an angle 60° with the horizontal. Find (*i*) the normal pressure, (*ii*) the frictional force and (*iii*) the resultant reaction of the plane on the body.

What is the magnitude of the normal pressure when the force acts downwards making the same angle with the plane.

[*Ans.* 11 lb. ; $3\sqrt{3}$ lb. ; $2\sqrt{37}$ lb. ; 29 lb.]

5. An object which weighs 180 gm. rests on a smooth inclined plane, inclined to the horizontal at an angle $\sin^{-1}\frac{3}{5}$. It is maintained in equilibrium by a string which makes with a line of greatest slope of the plane (and above it) an angle $\sin^{-1}\frac{5}{13}$. Find the tension in the string and the reaction of the plane. *Ans.* 117 gm. wt. ; 99 gm. wt.

6. Two forces, P, Q acting respectively parallel to the base and length of a smooth inclined plane, would each of them singly support a particle of weight W, prove that

$$W = PQ/\sqrt{P^2 - Q^2}.$$

7. Two weights P and Q rest one on each of two smooth planes placed back to back, of inclinations α and β, being connected by a string which runs horizontally from one to the other. Show that

$$P \tan \alpha = Q \tan \beta.$$

If the string passes over a smooth pulley at the top of the inclined planes, show that

$$P \sin \alpha = Q \sin \beta.$$

8. Show that the smallest force which will keep a body in equilibrium on a smooth inclined plane must act along the plane.

9. In what direction is the algebraic sum of the resolved parts of two forces acting at a point the greatest and in what direction is it the least ? Give reasons.

Ans. In the direction of the resultant ; perp. to it.

2.5. Equilibrium of particles connected by strings. When two or more particles are connected together, the equilibrium of each particle is considered separately, taking account of all the forces which arise in virtue of its connection (by strings or rods) with the others in addition to the special forces acting upon it.

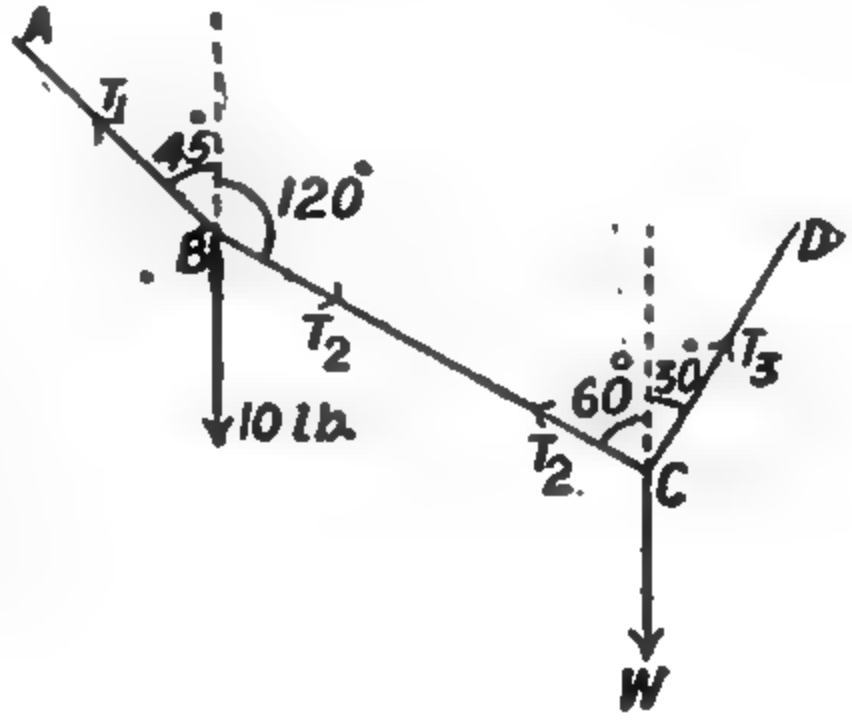
When there are only three forces acting on each particle, the conditions of equilibrium are easily obtained by the application of Lami's Theorem.

Examples VI.

1. *ABCD is a string suspended from points A and D and it carries a weight of 10 lb. at B and a weight of W lb. at C. The inclinations to the vertical of AB and CD are 45° and 30° respectively, and the angle ABC is 165°. Find W and the tensions in the different parts of the string.*

Let the tensions in the strings AB, BC and CD be T_1, T_2, T_3 respectively.

The string BC pulls B towards, C with the force T_2 and it pulls C towards B with the same force, the tensions T_2 being equal and opposite on B and C.



Since the forces T_1, T_2 and 10 lb., acting at B, are in equilibrium, we have

$$\frac{T_1}{\sin 60^\circ} = \frac{T_2}{\sin 45^\circ} = \frac{10}{\sin 165^\circ}$$

so that $$T_1 = \frac{10 \sin 60^\circ}{\sin 15^\circ} = 10 \times \frac{\sqrt{3}}{2} \times \frac{2\sqrt{2}}{\sqrt{3}-1} = 5\sqrt{6}(\sqrt{3}+1) \text{ lb.}$$

and $$T_2 = \frac{10 \sin 45^\circ}{\sin 15^\circ} = \frac{10}{\sqrt{2}} \times \frac{2\sqrt{2}}{\sqrt{3}-1} = 10(\sqrt{3}+1) = 27.32 \text{ lb.}$$

Again, since the forces T_2, T_3 and W, acting at C, are in equilibrium, we have

$$\frac{T_2}{\sin 30^\circ} = \frac{T_3}{\sin 60^\circ} = \frac{W}{\sin 90^\circ}$$

$$\therefore \quad T_3 = \frac{\sin 60^\circ}{\sin 30^\circ} . T_2 = \sqrt{3} \times 10(\sqrt{3}+1) = 10(3+\sqrt{3}) = 47.32 \text{ lb.}$$

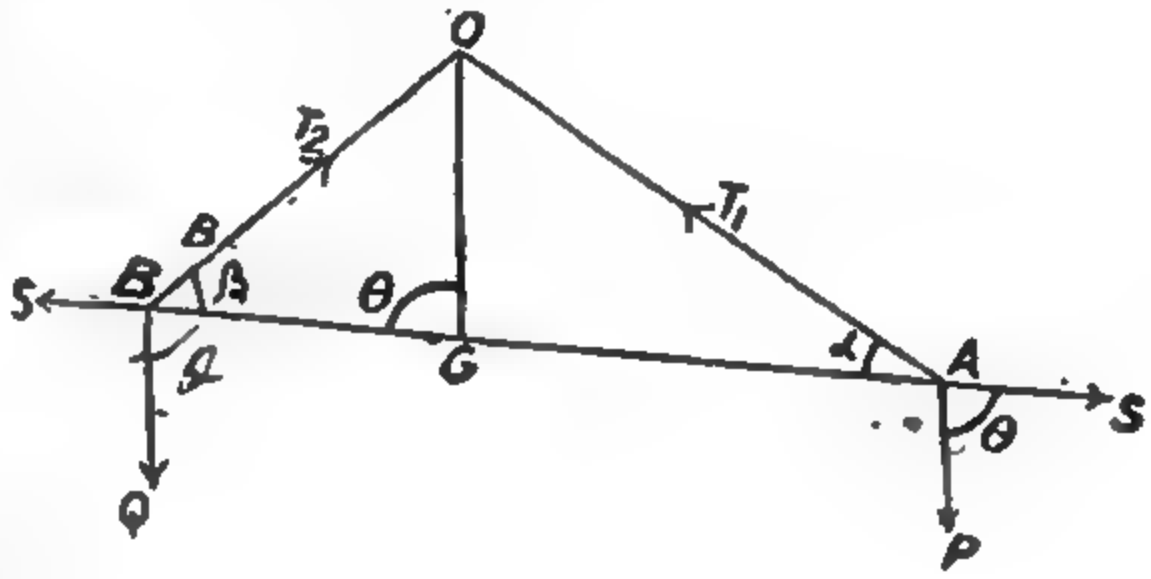
and $$W = \frac{\sin 90^\circ}{\sin 30^\circ} . T_2 = 2T_2 = 20(\sqrt{3}+1) = 54.64 \text{ lb.}$$

2. *Two weights P and Q are suspended from a fixed point O by strings OA, OB and are kept apart by a* **light** *rod AB. If*

the strings make angles α and β with the rod, show that the angle θ which the rod makes with the vertical is given by.

$$(P+Q)\cot\theta = P\cot\alpha - Q\cot\beta.$$

The points A and B being connected by a light rod AB, the thrust S of the rod AB at A is equal and opposite to that at B.



Let OG be the vertical through O meeting the rod in G, then $\angle OGB = \theta$.

Since forces T_1, the tension in the string AO, S the thrust of the rod and the weight P, acting at A, are in equilibrium we have

$$\frac{S}{\sin(\alpha+\pi-\theta)} = \frac{P}{\sin\alpha} = \frac{T_1}{\sin\theta} \quad \ldots\ldots(1)$$

and since the forces T_2, the tension in the string BO, S, the thrust of the rod and the weight Q, acting at B are in equilibrium we have

$$\frac{S}{\sin(\theta+\beta)} = \frac{Q}{\sin\beta} = \frac{T_2}{\sin\theta,} \quad \ldots\ldots(2)$$

$\therefore$ equating the values of S from (1) and (2), we get

$$\frac{Q\sin(\theta+\beta)}{\sin\beta} = \frac{P\sin(\alpha+\pi-\theta)}{\sin\alpha} = \frac{P\sin(\theta-\alpha)}{\sin\alpha}$$

i.e, $Q(\sin\theta\cot\beta+\cos\theta) = P(\sin\theta\cot\alpha-\cos\theta)$

which gives $(P+Q)\cot\theta = P\cot\alpha - Q\cot\beta.$

Note. (i) Triangles AOG and BOG are triangles of forces for the equilibrium of points A and B.

$$\therefore \frac{S}{AG} = \frac{P}{OG} \text{ and } \frac{S}{BG} = \frac{Q}{OG} \quad \therefore P\times AG = Q\times BG$$

(ii) Tensions T_1 T_2 can be obtained, if required from (1) and (2).

3. ABCD is a string suspended from points A and D and it carries a weight of 10 lb. at B and a weight W lb. at C. The portion AB is horizontal and BC, CD make angles of 30° and 60° respectively with the horizontal ; find W and the tensions in the three portions of the string.

[*Ans.* W=20 lb., tensions $10\sqrt{3}$, 20, $20\sqrt{3}$ lb.

4. A string ABCD attached to two fixed points A and D, has two equal weights, W, knotted to it at B and C and rests with the portions AB and CD inclined at angles of 30° and 60° respectively to the vertical. Find the tensions of the portions of the string and the inclination of BC to the vertical.

[*Ans.* $W\sqrt{3}$, W, W ; 60°.

5. ABCD is a light string to which a weight 12 lb. is attached at B and a weight W at C. The ends A, D are fastened to fixed points and the parts AB, BC, CD make angles 30°, 45°, 60° with the downward vertical measured in the same sense. Prove that $W=4\sqrt{3}$ lb. and find the tensions in the different parts of the string.

[*Ans* AB, 328 lb. wt : BC, 232 lb. ; CD, 189 lb. wt.

6. Three smooth pegs A, B, C are fixed in a vertical wall. The $\triangle$ABC is isosceles (AB=AC). A is uppermost and BC is horizontal, the angle BAC being α.

A light inextensible string carrying equal weights W at its ends is hung over the pegs. Prove that the thrust of the string on peg B is $2W \sin \frac{\alpha}{4}$ and find the thrust on the peg A.

[*Ans.* $2W \cos \frac{\alpha}{2}$.

7. A square board ABCD, the length of whose side is a, is fixed in a vertical plane with two of its sides horizontal. An endless string of length l ($>4a$) passes over four pegs at the angles of the board and through a ring of weight W, which is hanging freely. Show that the tension of the string is

$$W(l-3a) \;/\; 2\sqrt{l^2-6al+3a^2}.$$

8. Three equal strings are knotted together to form an equilateral triangle ABC and a weight W is suspended from A.

If the triangle and weight be supported with BC horizontal, by means of two strings at B and C, each at the angle of 135° with BC, show that the tension in BC is W $(3-\sqrt{3})/6$.

9. A light rod AB hangs horizontally by two strings from a point C and weights 8 lb. and W lb. are attached to A and B respectively. If the angle ABC=60° and the tensions in the string AC is 10 lb. wt., find (*i*) the value of W (*ii*) the tension in the string BC, (*iii*) the angle BAC and (*iv*) the stress in the rod. [*Ans.* $6\sqrt{3}$ lb. ; 12 lb. ; $\sin^{-1}\frac{4}{5}$; 6 lb.

10. A light rod AB, 5 ft. long, is hung from a peg O by two light strings OA, OB which are 4 ft. and 3 ft. long respectively. A weight W_1 is fixed to the rod at A and a weight W_2 at B. If in the position of equilibrium, the angle between the rod and the vertical is θ, show that

$$\tan\theta = \frac{12\,(W_1+W_2)}{16W_1-9W_2}.$$

11. A light rod AB of length .. in. hung from a fixed point O by two light strings OA, OB of lengths 4 and 3 in. respectively. A particle of weight 3 lb. is attached to the rod at A, and another of weight 2 lb. at B. Show that in the position of equilibrium the tensions in the strings OA, OB are $2\sqrt{5}$, $\sqrt{5}$ lb. wt. respectively and that the stress in the rod AB is $\sqrt{5}$ lb

12. A thin smooth circular wire, of ,radius a, is fixed in a vertical plane, and two small rings A, B of weights W, 2W are connected by a light rod AB of length $a\sqrt{3}$ and can slide freely on the wire. Show that in the position of equilibrium (below the centre of the wire) the rod is inclined at an angle of 60° to the vertical and that the stress in the rod is equal to 2W.

***13.** A light rod BC of length a, is suspended from a fixed point A by means of two light strings AB and AC of lengths c and b respectively. If masses each of weight W are suspended at B and C, show that, in the position of equilibrium, the tensions of the strings are proportional to their lengths and that the thrust in the rod is of magnitude $aW/\sqrt{2b^2+2c^2-a^2}$. (L U.)

***14.** Two spheres are connected by a string passing over a smooth peg and hang in contact. Prove that the distances of their centres (not necessarily lying in a horizontal line) from the peg are inversely proportional to their weights. (B. U.)

***15.** Two weights P, Q attached to the ends of a string rest on a smooth circular cylinder whose axis is horizontal. Prove that the inclination θ to the horizontal of the line joining them is given by

$$\tan\theta = \frac{P-Q}{P+Q}\tan\alpha.$$

where 2α is the angle which this line subtends at the nearest point of the axis of the cylinder.

***16.** Two weights w, w' rest on a smooth sphere, being supported by a string, (of natural length a and modulus λ) which passes through a smooth ring at O, a point vertically above C, the centre of the sphere. Show that in the position of equilibrium we have

$$\lambda\frac{l-a}{a}.\,OC = \frac{l}{\frac{1}{w}+\frac{1}{w'}},$$

where l denotes the length of the string in the position of equilibrium. (Pub. Ser. Com.)

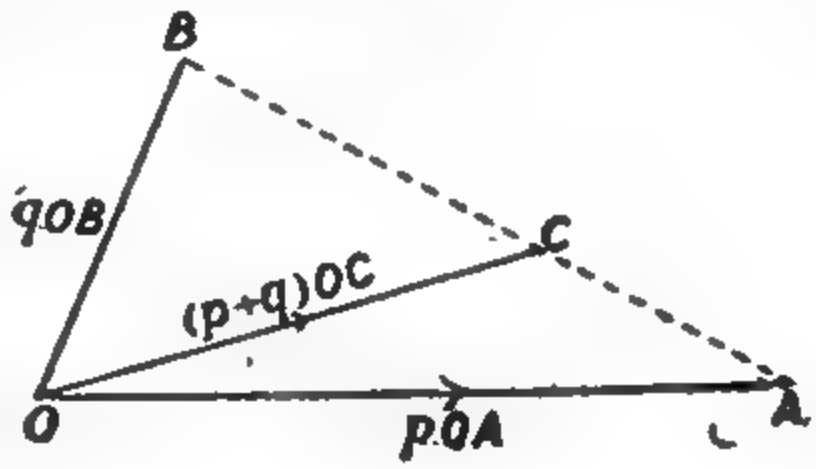
CHAPTER III

FORCES ACTING AT A POINT-II.

3.1. *The resultant of two forces acting at O, along OA and OB, represented in magnitude by p.OA a d q. OB, is represented by* $(p+q)$ *OC, where C is a point in AB such that* $p.CA = q.CB$.

By Art. 2 2, the force p.OA is equivalent to forces represented by p.OC and p.CA. Similarly the force q.OB is equivalent to forces represented by q OC and q.CB.

Hence the forces p.OA and q.OB are together equivalent to forces



represented by $(p+q)$ OC, p.CA and q.CB. But since p.CA$=q$ CB, the two latter forces being equal in magnitude and opposite in direction balance and we are left with only one force$(p+q)$ OC which is the resultant of the given forces.

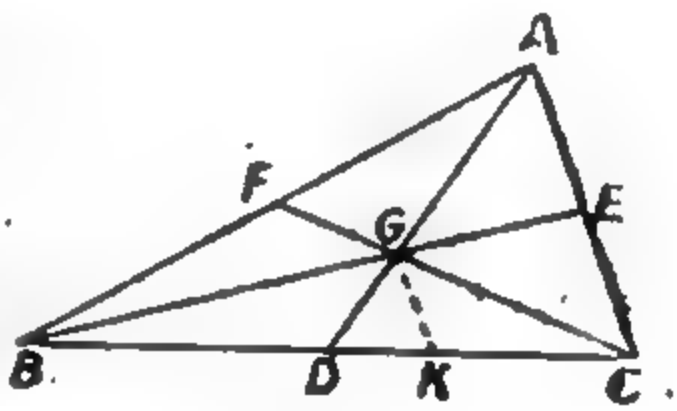
Cor. (i) *If the forces are p.AO and q.BO, the resultant is $(p+q)$CO.*

(ii) *If $p=q$, then C is the middle point of AB and the resultant is $2p$. OC.*

Note. If $p=-q$, the forces become q.AO and q.OB, their resultant is q. AB (Art. 2.2)

Example. (i) *If G is the point of intersection of the medians of a triangular lamina ABC, prove that forces acting at G and represented by* (a) *GA, GB, GC are in equilibrium, and* (b) *GA, 2GB, 3GC, have a resultant force represented by 3GK where K is a point on BC such that BK=2KC.*

(a) The resultant of forces GB and GC is 2GD where D is the middle point of BC.



2GD being equal to AG, the forces represented by 2GD and GA, being equal and opposite balance.

Hence *the forces acting at G and represented by GA, GB and GC are in equilibrium.*

(b) Now we are left with foress GB and 2GC acting along GB and GC. Their resultant, by the above article, is equal to 3GK where K is a point in GC such that BK=2KC.

Example (ii) *P is a point in the plane of the triangle ABC. Show that the resultant of the forces acting at P and represented by PA.sin A, PB.sin B and PC.sin C is*

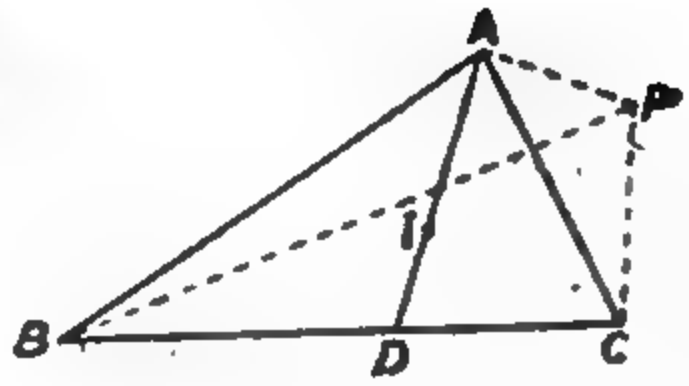
$$4PI.\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

where I is the incentre of the triangle ABC.

$y^2(1+x)^2 = x(1-x)$

The resultant of PB. sin B and PC. sin C is PD. (sinB+sin C), where D is a point on BC such that

$$\frac{BD}{DC}=\frac{\sin C}{\sin B}=\frac{c}{b},$$



i.e., AD bisects ∠A.

Now the resultant of PD. (sin B+sin C) and PA. sin A is PI. (sin A+sin B+sin C)

$$=4PI.\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\quad\text{(By Trign.)}$$

where I is a point on AD such that

$$\frac{AI}{DI}=\frac{\sin B+\sin C}{\sin A}=\frac{b+c}{a}$$ i.e., I is the incentre.

Examples VII.

1. Find a point inside a triangle such that if a particle at it be acted on by forces represented by the lines joining it to the vertices, it will be in equilibrium.

[*Ans.* The point of intersection of the medians.

2. ABC is a triangle ; D, E, F are the middle points of the sides BC, CA, AB respectively ; show that the forces represented by the straight lines AD, BE, CF, and acting at a point are in equilibrium. (D. U. 1937)

3. The sides AB and AC of a triangle ABC are bisected in D and E. Show that the resultant of forces represented by BE and DC is represented in magnitude and direction by $\frac{3}{2}$ BC.

4. O is any point in the plane of the triangle ABC and D, E, F are the middle points of the sides. Show that the system of forces OA, OB, OC is equivalent to the system OD, OE, OF.

5. Find a point O inside a quadrilateral ABCD such that

if a particle at it be acted upon by forces represented by OA, OB, OC, OD, it will be in equilibrium.

[*Ans.* The point of intersection of the lines joining the mid-points of the opposite sides.]

6. ABCD is a parallelogram, a particle P is attracted towards A and C by forces mPA and mPC respectively and repelled from B and D by forces nPB and nPD, show that the resultant passes through the centre O of the parallelogram.

Under what conditions is P in equilibrium wherever it is situated ?

[*Ans.* $m=n$.]

7. If forces P and Q act along OA, OB respectively, show that the line of action of their resultant cuts AB in a point C such that

$$AC/CB = Q.OA/P.OB.$$

8. Forces acting on a particle are represented by AP, PB, PC, where P is the orthocentre of $\triangle ABC$. Show that the resultant is represented in magnitude and direction by the diameter through A of the circle ABC.

9. H is the orthocentre and O the circumcentre of a triangle ABC. Show that the three forces AH, BH, CH, have as resultant the force represented by twice OH.

10. Points X, Y, Z are taken on the sides BC, CA, AB of a triangle ABC such that

$$\frac{BX}{XC}=\frac{CY}{YA}=\frac{AZ}{ZB}=\frac{l}{m}.$$

Prove that forces acting at a point and represented in magnitude and direction by AX, BY and CZ are in equilibrium.

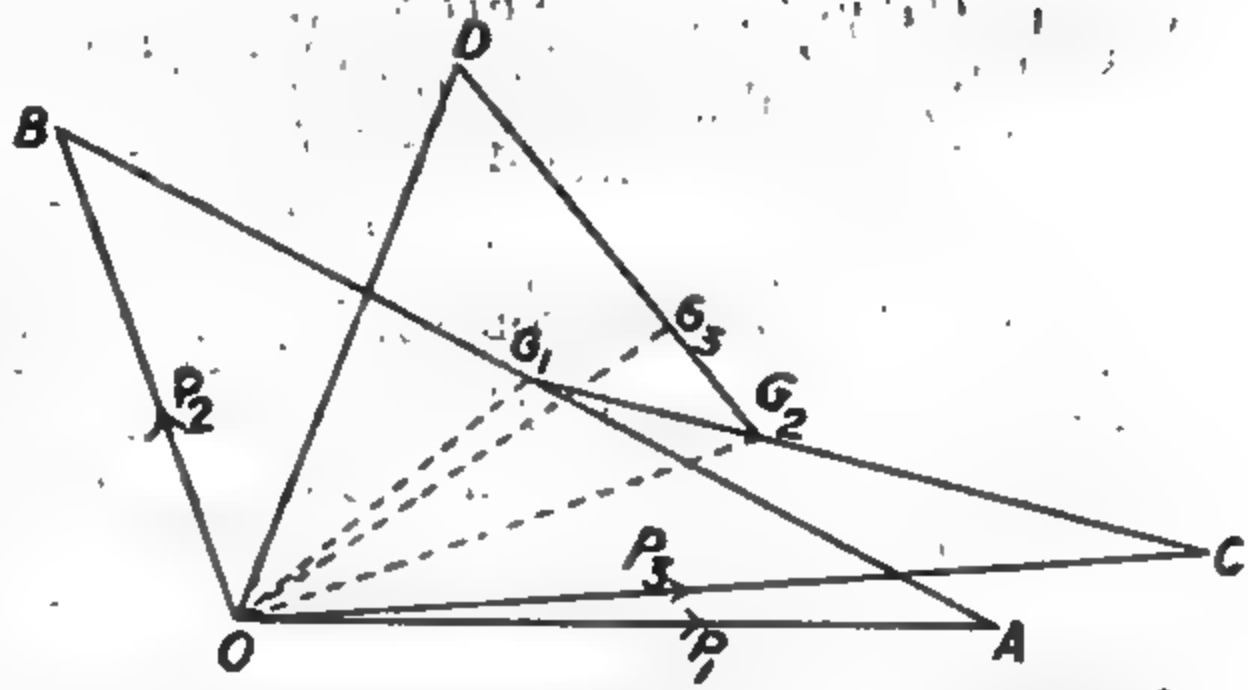
Resultant of a number of concurrent forces.

3.2. The resultant of a number of forces *acting at a point* can be obtained in the following ways :—

3.21. **First Method.** *By the repeated application of Art.* 3·1.

Let the forces P_1, P_2, P_3.....etc. acting at O be repre-

sented by the lines OA, OB, OC...etc.



The resultant of P_1 and P_2 is $2OG_1$, where G_1 is the middle point of AB. Compounding the force $2OG_1$ with P_3 (OC) we get a resultant represented by $3OG_2$, where G_2 divides G_1C such that $2G_1G_2 = G_2C$. Again, the resultant of $3OG_2$ and P_4 (OD) is $4OG_3$ where $3G_2G_3 = G_3D$ and so on.

Proceeding in this way, we obtain the resultant of n such forces as $n.OG$ where G is the last point arrived at in this manner.

Def. The point G is known as the centroid of points A, B, C, D...etc.

Cor. *If the point G so obtained happens to coincide with O, the resultant vanishes, and the forces are in equilibrium.*

In other words, the forces are in equilibrium, if the point O at which the forces act is itself the centroid of the extremeties of the vectors representing the forces.

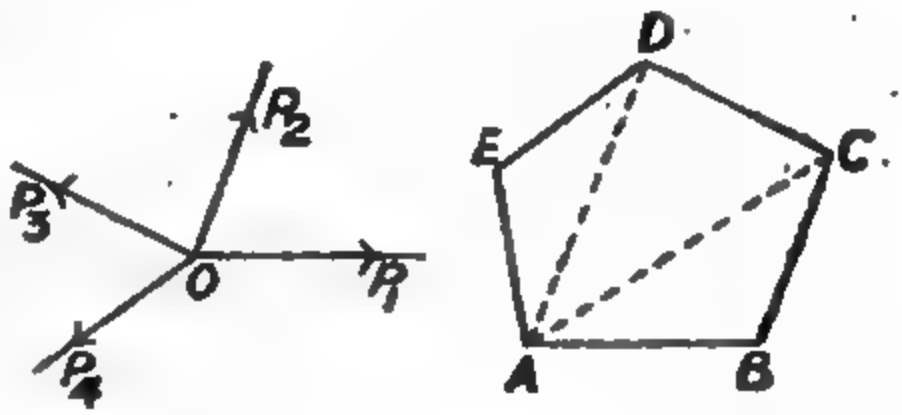
Note It is clear from the proof that the forces need not necessarily lie in the same plane.

Example. If A, B, C, D.....are points which divide the circumference of a circle into n equal parts and if a particle P, lying on the circumference is acted upon by forces represented by PA, PB, PC, PD..., the magnitude of the resultant is constant.

For, by symmetry the centre O of the circle is the centroid of A, B, C, D.....and the resultant of the forces is, therefore, equal to n. $PO = nr$, a constant, r being the radius of the circle.

3.22. Second Method. *By the repeated application of the law of parallelogram of forces.* The process, in its simplified form, is as follows :—

Let forces P_1, P_2, P_3, act at the point O. Let AB and BC represent the forces P_1 and P_2, their resultant, say, R_1, is then represented by AC (Art. 2.2).



Now let CD represent P_3, the resultant R_2, of R_1 and P_3 is represented by AD, *i.e.*, the resultant of P_1, P_2, P_3 is represented by AD. Similarly if DE represents the force P_4, AE will respreeent the resultant of forces P_1, P_2, P_3 and P_4.

The process can be continued in the case of a number of forces acting at a point, the final resultant being represented by the line joining the first point A to the end of the line representing the last force.

Cor. *When the end of the last line coincides with the first point, the resultant vanishes.*

This leads to the theorem of the polygon of forces given below.

3 221. Polygon of Forces. *If a number of forces acting at a point can be represented by the sides of a closed polygon, taken in order, the forces are in equilibrium.*

Let AB, BC, CD, DE and EA represent, on some scale. the forces P_1, P_2, P_3, P_4 and P_5. As shown above the resultant of forces AB, BC, CD and DE is represented, in magnitude and direction, by AE ; this resultant together with the force P_5 represented by EA, balance. Hence the forces are in equilibrium.

This applies to any number of forces and proves the theorem.

Note. It is clear from the proof that the forces need not necessarily be coplanar, and the sides of the polygon need not lie in the same plane.

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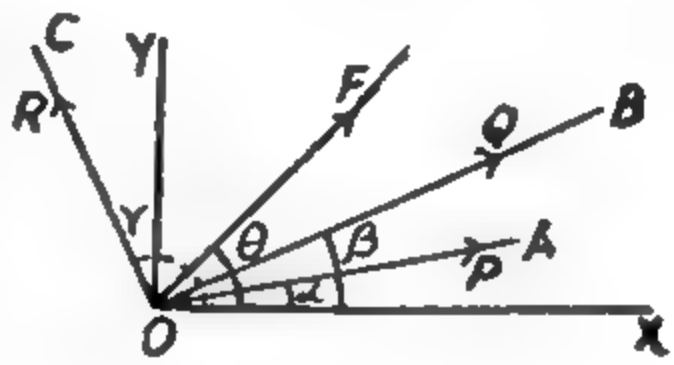
3.222. Converse of the Polygon of forces. If a number of forces, acting at a point are in equilibrium, they can be represented by the sides of a closed polygon taken in order, the sides being respectively parallel to the direction of the forces.

This proposition is not complete in the sense in which the converse of the triangle of forces is. For, in the case of three forces, all triangles having their sides parallel to the forces are similar and consequently, the sides of each one of them can represent the forces on some scale, while in the case of a number of forces, all polygons having their sides parallel to the forces are not necessarily similar (as can be seen by drawing a line D'E' parallel to DE in the figure of Art. 3.22) and the sides of all these polygons do not therefore represent the given forces.

3.23. Third Method. Analytical method for determining the resultant of concurrent and coplanar forces.

Let OX, OY be two perpendicular lines in the plane of the forces.

Let the forces P, Q, R...etc. act along OA, OB, OC etc., making angles α, β, γ...etc., with OX, let F be their resultant inclined at $\angle\theta$ with OX and let X, Y be the resolved parts of F along OX and OY.



Then, since the resolved part of the resultant is equal to the sum of the resolved parts of its components (Art. 2 4), by resolving forces along OX, and OY, we have

$$F\cos\theta = P\cos\alpha + Q\cos\beta + R\cos\gamma + \ldots = X \quad \ldots\ldots(1)$$

and $$F\sin\theta = P\sin\alpha + Q\sin\beta + R\sin\gamma + \ldots = Y \quad \ldots\ldots(2)$$

$$\therefore \quad F^2 = X^2 + Y^2 \qquad (3)$$

$$= \Sigma P^2 + 2\Sigma PQ\cos(\alpha - \beta).$$

and $$\tan\theta = \frac{Y}{X} \qquad \ldots\ldots(4)$$

Equations (3) and (4) give the magnitude and line of action of the resultant.

3.231. When $X=0$ as well as $Y=0$, then $F=0$ so that the forces are in equilibrium if the sum of their resolved parts along two perpendicular lines, OX, OY in their plane, separately vanish.

Conversely, if the forces are in equilibrium, $F=0$. It follows from (3) that $X=0$ and also $Y=0$.

These conditions are, therefore, *sufficient as well as necessary conditions of equilibrium* for concurrent and co-planar forces.

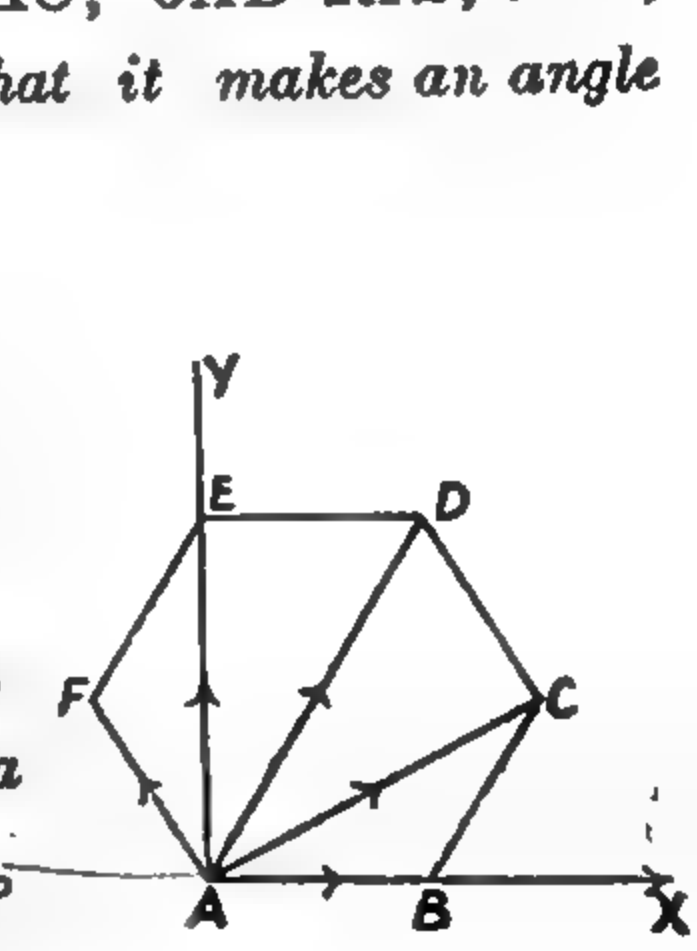
3.232. It may be noted that it is simply for the sake of convenience that OX, OY are taken to be at right angles, it is not necessary to do so.

For forces to be in equilibrium, it is sufficient that the sum of the resolved parts of the forces along any two straight lines separately vanish. For, if the sum of the resolved parts along one line l vanishes, the resultant F is either zero or acts at right angles to l. Similarly if the sum of the resolved parts along any other line l' vanishes, F is either zero or acts at right angles to l'. Hence as F cannot be perpendicular to two different lines l and l', it must be zero.

Example (i) *ABCDEF is a regular hexagon. Show that the rssultant of forces represented by AB, 2AC, 3AD 4AE, 5AF, is represented by* $\sqrt{351}$ *AB and show that it makes an angle* $\tan^{-1}\frac{7}{\sqrt{3}}$ *with AB.*

Taking a to be the length of a side of the hexagon and resolving along AB and AE, two lines at right angles, we have

$$X=AB+2AC\cos 30°+3AD\cos 60°+5AF\cos 120°$$
$$=a+2\sqrt{3}a\times\sqrt{3}/2+6a\times\tfrac{1}{2}-5a\times\tfrac{1}{2}=\tfrac{9}{2}a$$
$$Y=2AC\sin 30°+3AD\sin 60°+4AE+5AF\cos 30°$$



$$=2\sqrt{3}a\times\tfrac{1}{2}+6a\times\frac{\sqrt{3}}{2}+4\sqrt{3}a+5a\times\frac{\sqrt{3}}{2}$$

$$=\frac{2\sqrt{3}}{2}\,a$$

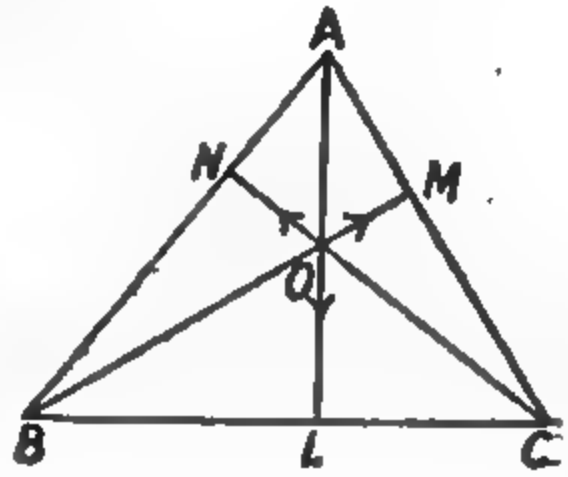
$\therefore$ the resultant $R=\sqrt{X^2+Y^2}=\sqrt{35.1a}=\sqrt{35.1}AB$

and $$\tan\theta=\frac{Y}{X}=\frac{7}{\sqrt{3}}.$$

Example (*ii*) *Forces act through the angular points of a triangle perpendicular to the opposite sides and are proportional to the cosines of the corresponding angles, show that the resultant F is proportional to*

$$\sqrt{1-8\cos A\cos B\cos C}.$$

Let the forces along AL, BM. CN be equal to $k\cos A$, $k\cos B$, $k\cos C$, respectively. Resolving along LC taken as X-axis, and LA as Y−axis, we have



$$X=(k\cos B)\sin C-(k\cos C)\sin B$$
$$=k\sin(C-B).$$

[$\because$ $\angle LBM=90°-C$, $\angle LCN=90°-B$

$$Y=(k\cos B)\cos C+(k\cos C)\cos B-k\cos A$$
$$=2k\cos B\cos C-k\cos A$$
$$=k[\cos(C-B)+\cos(C+B)]-k\cos A$$
$$=k\cos(C-B)-2k\cos A \qquad \because\ B+C=180°-A$$

$\therefore$ $F^2=X^2+Y^2=k^2[1+4\cos^2 A-4\cos A\cos(C-B)$
$$=k^2-4k^2\cos A[\cos(C+B)+\cos(C-B)]$$
$$=k^2(1-8\cos A\cos B\cos C).$$

EXAMPLES VIII

1. Find the magnitude and direction of the resultant of three coplanar forces P, 2P, 3P acting on a particle and inclined mutually at an angle of 120°.

[*Ans.* $\sqrt{3}$P perp. to 2P.]

2. Forces 11, 4 and 10 lb. wt. act along OA, OB and OC respectively. These lines are in the same plane such that $\angle AOB = 90°$ and $\angle BOC = \sin^{-1}\frac{3}{5}$. Find the magnitude and direction of the resultant. [*Ans.* 13 lb. at 67° 23′ with OA.

3. ABCD is a rectangle having AB = 4″ and BC = 3″; forces 2 lb. wt. along AB, $6\frac{2}{5}$ lb. wt. along AD and 8 lb. wt. along the diagonal AC, all act at the point A. Find the resultant.

[*Ans.* 14 lb. wt. inclined with AB at an angle = ∠CAD.

4. Four horizontal wires attached to a telephone post exert the following tensions on it. 20 lb. wt. N., 30 lb. wt. E., $40\sqrt{2}$ lb. wt. S. W. and $50\sqrt{2}$ lb. wt. S. E. Calculate the resultant pull on the post and find its direction.

[*Ans.* $R = 10\sqrt{65}$ lb. wt. at an angle $\tan^{-1}\frac{7}{4}$ S. of E.

5. A string of length 31 in. has its ends tied to two points in a horizontal line at a distance 25 inches apart. A small ring from which is suspended a weight of 9 lb. can slide on the string, and is acted on by a horizontal force of such magnitude that in the position of equilibrium the ring is at a distance 7 inches from the nearer end of the string. Show that the force is approximately 5 lb. wt. and find the tension of the string.

[*Ans.* $7\frac{8}{31}$ lb. wt.

6. The sides of a quadrilateral are 1, 3, 5, 6. Forces of 2, 6, 9 and 12 lb. wt. respectively act *on a particle* parallel to the sides of the quadrilateral taken in order. Find the resultant. [*Ans.* 1 lb. wt.

7. A, B, P, Q, R are five points on a plane. Forces AP, AQ, AR act at A, and forces PB, QB, RB at B. Prove that the resultant of the six forces is 3AB in magnitude and direction. (All. U.)

8. Forces each equal to P act at a point, parallel to the sides of the triangle ABC, taken in the same sense, show that their resultant R is given by

$$R^2 = P^2(3 - 2\cos A - 2\cos B - 2\cos C)$$

9. ABC is an equilateral triangle inscribed in a circle. Repelling forces P, Q, R directed from the points A, B, C res-

pectively, act on a particle P situated on the circle midway between B and C. Show that the resultant force on P is

$$\left[P^2+Q^2+R^2-(QR+RP+PQ)\right]^{\frac{1}{2}}$$

10. Show that if $(2n+1)$ co-planar forces each of magnitude P act at a point, the magnitude of their resultant cannot be greater than $(2n+1)$ P, and find the least value of their resultant. [*Ans.* zero.

11. Forces 1, 2, 3, 4, 5, 6 lb. wt. act at a point in the directions of successive sides of a regular hexagon taken in order. Show that the resultant is a force of 6 lb. wt. acting in the direction of the 5 lb. wt. force.

12. ABCDEF is a regular hexagon. Find the resultant of forces represented by AB, AC, AD, AE and AF.

[*Ans.* 6AB acting along AD.

13. A string of length l is fastened to two points A, B at the same level, at a distance a apart. A ring of weight W can slide on the string and a horizontal force X is applied to it such that it is in equilibrium vertically below B. Prove that $X=\frac{a}{l}W$ and that the tension of the string is

$$W.\frac{a^2+l^2}{2l^2}.$$

3·3 Conditions of Equilibrium of Forces acting on a particle.

The conditions for the *equilibrium of a particle* subject to two or more forces may be summarised as follows :—

(*a*) *Two forces* acting on a particle are in equilibrium if they are equal and opposite and collinear in action.

(*b*) *Three forces* acting on a particle are in equilibrium if they lie in the same plane and satisfy Lami's Theorem. (Art. 2·3)

Or

if the sum of their resolved parts along two lines in their plane separately vanish. (Art. 3·231).

Or

if the forces can be represented by the sides of a triangle taken in order. (Art. 2·21).

(c) *A number of forces* acting on a particle are in equilibrium.

if they can be represented by the sides of a closed polygon. (Art. 3·221).

Or

if the point O at which the given forces act is the centriod of the extremities of lines representing the forces. (Art. 3·21).

Or

if the forces are co-planar and the sum of their resolved parts along two straight lines in their plane separately vanish. (Art. 3·231).

The conditions given above are necessary as well as sufficient.

3·4. Equilibrium of three non-parallel forces acting on a rigid body. *If three* **co-planar** *and non-parallel forces acting on rigid body. keep it in equilibrium, the forces must meet at a point.*

Let P, Q, R be the three forces. Since they are co-planar and not all parallel at least two of them, say P and Q, must meet at some point O and the resultant of P and Q is a force also passing through O. Since the three forces are in equilibrium, the third force R, must be equal, opposite and collinear in action with the resultant of P and Q, so that R also must pass through O.

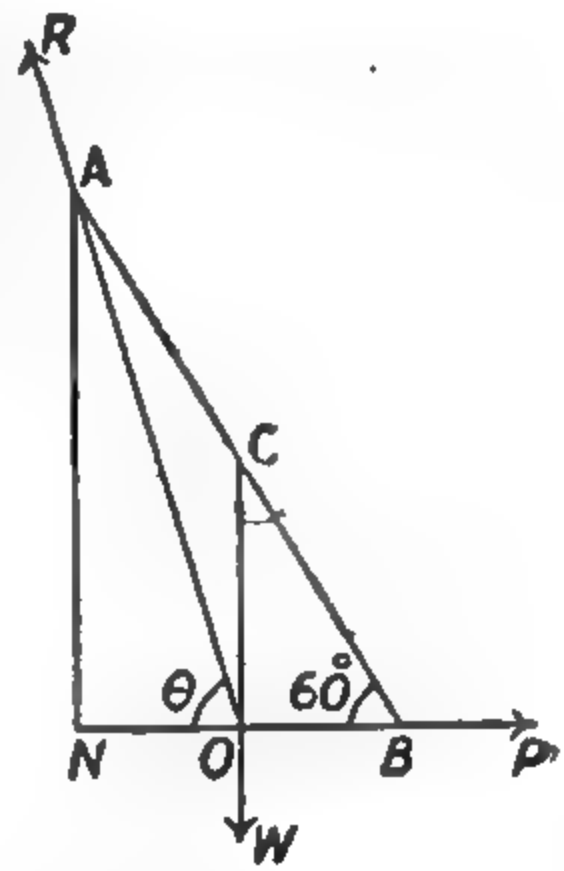
The forces being concurrent, the conditions of equilibrium as given in Art. 3·3 (*b*) are applicable.

Note—1. Three forces keeping a rigid body in equilibrium must necessarily be co-planar. For proof see Art. 6·3.

Note—2. A few easy examples involving three non-parallel forces are given below. Other examples of such forces acting on bodies will be given in Arts. 4·65 and 6·31.

Example. (*i*). *A uniform rod AB of weight W lb. is freely hinged at A and is supported so that it makes an angle of 30° with the downward vertical by a horizontal force P at B. Find P and the action at the hinge.*

Let the vertical through C, the middle point of the rod, meet the horizontal line of action of P in O.



The forces acting on the rod are its weight W, the force P, and R the reaction of the hinge.

Since the weight of the rod acting at C and the force P meet in O, therefore the third force R must also pass through O and must act along the line OA.

Let OA make an angle θ with the horizontal, then

$$\tan\theta = \frac{NA}{NO} = \frac{NA}{\frac{1}{2}NB} = 2\frac{NA}{NB} = 2\tan 60° = 2\sqrt{3}.$$

Now resolving forces vertically and horizontally, we have

$$R\sin\theta = W \text{ i.e., } R = W\operatorname{cosec}\theta = W\sqrt{1+\cot^2\theta}$$
$$= W\sqrt{\tfrac{13}{12}}.$$

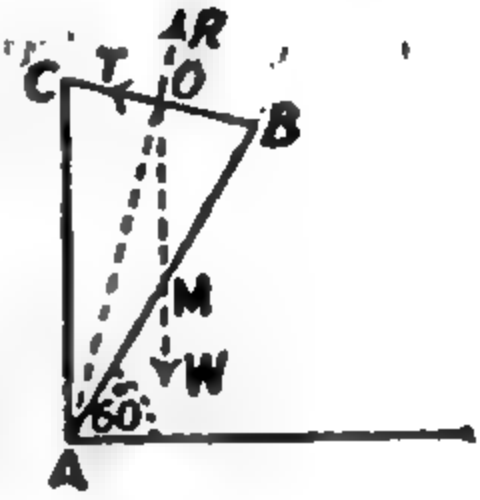
$$\text{and } P = R\cos\theta = \frac{W}{\sin\theta}\cos\theta = W\cot\theta = \frac{W}{2\sqrt{3}} = \tfrac{1}{6}W\sqrt{3}.$$

Note—Thes results can also be obtained by the application of Lami's Theorem or from the triangle of forces OAN.

Example. (*ii*). *A uniform beam AB of weight W can turn in a vertical plane about a hinge A, and to the other end B is tied a rope which is fastened to a point C, vertically above A, so that AC = AB. Find the tension in the rope necessary to keep the beam at an angle of 60° above the horizontal. Find also the direction and magnitude of the reaction of the hinge.*

The beam AB is in equilibrium under the action of three forces :—

(i) the tension T along BC.
(ii) the reaction R at the hinge A.
(iii) its weight W, acting along the vertical line through the middle point M.



Since two of these forces W and T meet in O (middle point of BC) the third force R must also pass through O, so that R acts along AO.

Again, since AB = AC and ∠BAC 30°,

∴ ∠ACB = ∠ABC = 75° and AO is at right angles to BC.

Hence ∠CAO = 15°, *i. e.*, the reaction acts in a direction making ∠15° with the vertical.

By Lami's Theorem we have

$$\frac{T}{\sin 15^\circ} = \frac{R}{\sin 75^\circ} = \frac{W}{\sin 90^\circ}$$

i. e., $T = W \sin 15^\circ$ and $R = W \cos 15^\circ$.

Note.—These results can also be obtained either from the traingle of force AOC or by resolving forces along AO and BC.

EXAMPLES IX

1. A heavy uniform rod AB of weight 2 lb is free to turn about its upper end A which is fixed and the rod is kept in equilibrium inclined at an angle of 30° to the vertical by a horizontal force P applied to the lower end B. Find P and the reaction at A. [*Ans.* $\frac{1}{3}\sqrt{3}$; $\frac{1}{3}\sqrt{39}$ lb. wt.

2. A uniform straight rod of mass 2 lb. is suspended freely from one end, and held in an oblique position by a horizontal force of 1 lb. wt. applied at the lower end. Find the inclination of the rod to the vertical.

Find also the magnitude and direction of the reaction at the point of suspension.

Ans. 45° ; 2·24 lb. wt. at $\tan^{-1} 2 = 63^\circ\ 26'$ to the horizontal.

3. A uniform bar AB of weight W and length *l* is hinged at its upper end A and a horizontal force is applied to the end

B so that the bar is in equilibrium with B at a distance a from the vertical through A. Show that the reaction at the hinge is

$$\tfrac{1}{2} W \left(\frac{4l^2-3a^2}{l^2-a^2}\right)^{\frac{1}{2}}$$

4. A uniform rod AB of mass 10 lb. is freely hinged about the end A, which is fixed, and is supported in an inclined position by a light string joining the end B to a fixed point C, vertically above A. If AB=AC=2BC, find completely the action of the hinge on the rod and the tension in the string.

[*Ans.* 9.68 lb. wt. bisecting angle BAC ; 25 lb. wt.

5. Two planes are inclined at angles α and β to the horizon and their line of intersection is horizontal. A sphere of weight W is placed between them, show that pressures on the planes are

$$\frac{W \sin \beta}{\sin (\alpha+\beta)} \quad \text{and} \quad \frac{W \sin \alpha}{\sin (\alpha+\beta)}$$

If $\alpha=60°$ and the pressure on this plane is $\frac{1}{2}$ W, show that $\beta=30°$

6. If a uniform heavy rod be supported by a string fastened at its ends and passing over a smooth peg, prove that it can rest only in a horizontal or vertical position.

In case the strings attached to the two ends are different show that the tensions along them are propotional to their lengths.

7. A uniform rod of weight 30 lb. and of length 10 ft. is supported in equilibrium by means of light inextensible strings of 6 ft. and 8 ft. whose ends are attached to the ends of the rod and to a fixed point. Find the tension in the strings.

[*Ans.* 24, 18 lb. wt.

8. A heavy uniform beam is hung from a fixed point by two strings attached to its extremities. If the strings and the beams be as 7 : 9 : 8. show that the tensions of the strings and weight of the beam are as 7 : 9 : 14.

9. A non-uniform beam AB 6 ft. long is held by two ropes AC and BD in a horizontal position. AC and BD make angles of 60° and 45° with the horizontal. Find the position on the beam of the point where its weight W acts and find the tensions in the ropes.

[*Ans.* 3·8 ft. from B, $W(\sqrt{3}-1)$, $\frac{1}{2}W(\sqrt{6}-\sqrt{2})$.

10. A uniform heavy rod AB has the end A in contact with a smooth vertical wall, and one end of a string is fastened to the rod at a point C, such that $AC=\frac{1}{4}AB$, and the other end of the string is fastened to the wall vertically above A. Find the length of the string if the rod rests in a position inclined to the vertical. [*Ans.* Equal to AC.

What happens when $AC=\frac{1}{2}AB$?

11. A uniform rod AB of length a, hangs with one end A against a smooth vertical wall, being supported by a string of length l, attached to the other end of the rod and to a point D of the wall vertically above A. Show that if the rod rests inclined to the wall at an angle θ,

$$\cos^2\theta=\frac{l^2-a^2}{3a^2}.$$

Show that in order that equilibrium may be possible, $a<l<2a$.

12. A rod AB of weight W rests over a smooth horizontal rail at C with its lower end A in contact with a smooth vertical wall. If the rod is inclined at an angle θ to the vertical. show that the magnitude of the reactions at A and C are respectively equal to

W cot θ and W cosec θ.

13. A uniform beam 12 ft. long and weighing 50 lb. rests against a smooth horizontal rail, the point of contact being 3 ft. from the lower end of the beam and is kept in equilibrium in this position by a horizontal force F applied to the lower end. Show that the beam is inclined at 45° to the horizontal, and find the magnitude of the force F. (P. U. 1930).

[*Ans.* F=50 lb.

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14. A uniform beam of length $2a$, rests in equilibrium against a smooth vertical wall and upon a peg at a distance b from the wall. Show that the inclination of the beam to the vertical is

$$\sin^{-1}\left(\frac{b}{a}\right)^{\frac{1}{3}}$$

(P. U. 1933.)

15. A uniform rod of length $4a$ is placed across a smooth horizontal rail and rests with one end against a smooth vertical wall, the distance of which from the rail is a. Show that the angle which the rod makes with the horizontal is $\text{Sec}^{-1} 2^{\frac{1}{3}}$.

(D. U. 1938).

16. A uniform equilateral triangular lamina ABC of weight W can turn in a vertical plane about a hinge B ; it is supported with side AB horizontal, by a smooth prop at the middle point of BC. Find the pressure on the prop and the reaction at the hinge. [*Ans.* Each = W

17. Prove that a uniform rod cannot rest entirely within a smooth hemispherical bowl except in a horizontal position.

18. A *light* rod rests between a smooth vertical wall and a smooth floor and a point P of the rod is connected by a string to the corner where the wall and floor meet. Determine the position of the point P on the rod.

[*Ans.* the midole point. Why ?

19. A uniform rod AB of weight 5 lb. and length 1 ft. is hinged to a axed point A. An elastic cord of natural length 1 ft.. whose modulus of elasticity is 10 lb. wt. is attached at one end to B and at its other end to a fixed point C 1 ft. vertically above A Find the tension of the string and also its length and inclination to the vertical.

[*Ans* $3\frac{1}{3}$ lb. wt., $1\frac{1}{3}$ ft, $\cos^{-1}\frac{2}{3}$.

3·5. Problems involving Friction. When a body is at rest in contact with a rough surface and the reaction S (which is equal and opposite to the resultant of the other forces acting on the body) does not act along the normal at the point of contact but is inclined to the normal

then F, the tangential component of this reaction, is the *force of friction* and R, the normal component, is the *normal pressure*

If θ be the angle which the resultant S makes with the normal, then

$$S \cos \theta = R, \qquad S \sin \theta = F,$$

and

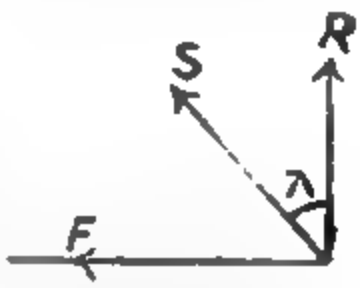
$$\tan \theta = \frac{F}{R}.$$

The resultant reaction S and consequently its components F and R will change with the change in the other forces acting on the body.

In the limiting case when the body is on the point of motion $\frac{F}{R}$ attains its greatest velue μ, the co-efficient of friction.

If λ be the angle which the resultant reaction in this case makes with the normal, we have

$$\tan \lambda = \frac{F}{R} = \mu$$



The angle λ is called the **angle of friction.** Thus we find that

the co-fficient of friction is equal to the tangent of the angle of friction.

It is evident that so long as the body does not commence to move $\theta < \lambda$ and $\frac{F}{R} < \mu$ *i.e.*, $F < \mu R$.

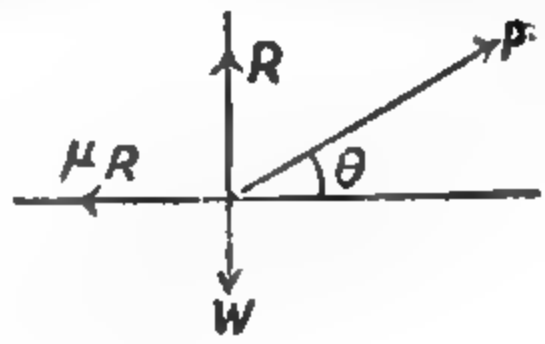
In other words, *the inclination of the resultant reaction to the normal cannot exceed a certain value* λ, *the angle of friction and the amount of friction F cannot bear more than a certain ratio* μ *to the normal pressure.*

The **Cone of friction** is the cone which has the point of contact as its vertex, the normal as its axis, and λ its semi-vertical angle.

In the case of limiting equilibrium the resultant reaction acts along a generator of this cone and when the equilibrium is not limiting the resultant reaction acts along a line lying within the cone.

3.6. *To find the least force to drag a particle on a rough horizontal plane.*

Let W be the weight of the body (regarded as a particle) and R, the normal pressure. Let P be a force acting at an angle θ with horizontal.



When the body is on the point of motion, there is limiting equilibrium and the force of friction is therefore μR or $R \tan\lambda$.

Resolving horizontally and vertically, we have.

$$P \cos\theta - \mu R = 0 \quad \ldots\ldots(1)$$

and $$P \sin\theta + R - W = 0 \quad \ldots\ldots(2)$$

Eliminating R, we get

$$P(\cos\theta + \mu \sin\theta) = \mu W.$$

$$P = \frac{\mu W}{\cos\theta + \mu \sin\theta} = \frac{W \sin\lambda}{\cos(\theta - \lambda)}.$$

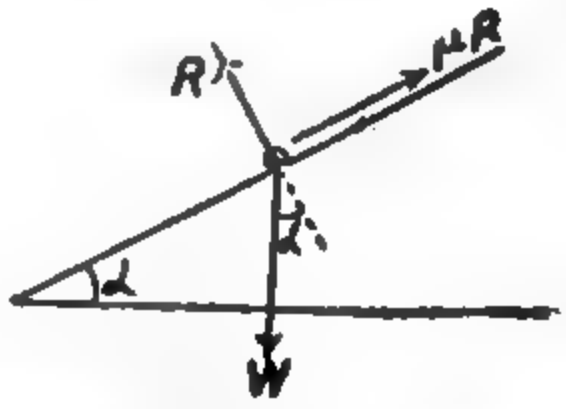
P is evidently least when $\cos(\theta - \lambda) = 1$ or $\theta = \lambda$ *i.e.*, the *force is least when it acts in a direction making with the horizontal, an angle equal to the angle of friction.*

Thus the least force to move a weight W along a rough horizontal plane is W sin λ, λ being the angle of friction.

3.7. *To find the greatest inclination of a rough plane on which a heavy particle can rest in equilibrium without the application of any other force.*

Let a body (treated as a particle) of weight W be placed on a rough plane and let the plane be tilted till the body is on the point of sliding down the plane.

Let α be the inclination of the plane in this position and R, the normal pressure.



Then the force of friction μR acts up the plane along the line of greatest slope through the body.

By resolving along and perpendicular to the plane, we have

$$W \sin\alpha = \mu R \quad \text{and} \quad W \cos\alpha = R$$

$$\therefore \qquad \tan \alpha = \mu R/R = \mu = \tan \lambda.$$

i.e., the *greatest inclination of the plane is equal to the angle of friction.*

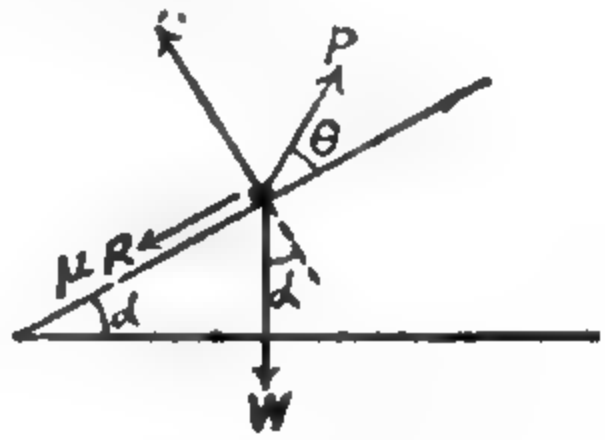
If $\alpha > \lambda$, the body slides down the plane and a force will be required to keep it at rest or to move it up the plane.

If $\alpha < \lambda$, a force will be necessary to move the body either up or down the plane.

3.8. *To find the direction and magnitude of the least force to drag a heavy particle up a given rough inclined plane.*

Let α be the inclination of the plane to the horizon, W the weight of the body. and R, the normal reaction.

Let P be a force acting at an angle θ with the line of greatest slope through the body.



Let the body be on the point of motion *up* the plane, so that the friction acts down the plane and is equal to μR.

Resolving along and pe pendicular to the plane, we have

$$P \cos \theta = \mu R + W \sin \alpha \qquad \ldots\ldots(1)$$

and

$$P \sin \theta + R = W \cos \alpha \qquad \ldots\ldots(2)$$

Eliminating R, we have

$$P(\cos\theta + \mu \sin \theta) = W(\sin \alpha + \mu \cos \alpha)$$

i.e.,

$$P = W \frac{(\sin \alpha + \mu \cos\alpha)}{\cos \theta + \mu \sin\theta} = W \frac{\sin (\alpha + \lambda)}{\cos (\theta - \lambda)} \qquad \ldots\ldots(3)$$

It gives the amount of force just sufficient to move the body up the plane and shows that force varies with θ, the angle at which the force is inclined to the plane.

P will evidently be least when cos $(\theta - \lambda)$ is the greatest *i.e*, when $\theta = \lambda$.

Hence the required force is inclined to the plane at an angle equal to the angle of friction and is of magnitude W sin $(\alpha + \lambda)$,

3.81. *To find the force required just to support the heavy particle* $(\alpha > \lambda)$. Let $P = P'$ when the body is on the point of motion *down* the plane.

In this case the force of friction acts up the plane and therefore by changing the sign of μ or λ in the above equations, we get

$$P' = \frac{W \sin(\alpha - \lambda)}{\cos(\theta + \lambda)} \quad \ldots\ldots(4)$$

which gives the amount of force just sufficient to support the body on the plane.

In this case the body needs support only when $\alpha > \lambda$ and as P′ ceases to act, the body moves down the plane.

Note If $\alpha < \lambda$, no force is needed to support the body for it will rest in equilibrium cn the plane. The least downward force required to make the body slide down the plane in this case can be obtained on similar lines.

Cor. *When $\alpha > \lambda$ and $\theta = 0$ i e., the force acts along the plane, then from* (3) *and* (4), *we have*

$$P = W.\frac{\sin(\alpha + \lambda)}{\cos \lambda} \text{ and } P' = W.\frac{\sin(\alpha - \lambda)}{\cos \lambda}.$$

The body will remain in equilibrium under the action of a force which lies between P and P′.

These results can be easily obtained independently.

EXAMPLES X

1. A small object of mass 10 lb. lies at rest on a rough horizontal plane, and a gradually increasing force P is applied to the particle in an upward direction making an angle of 30° with horizental ; it is found that the particle begins to move horizontally when P = 4 lb. wt. Show that the co-efficient of friction is about 0 433, and, if the force P acts in a downward direction at an angle of 30° to the horizontal, find the magnitude of P when the particle just begins to move.

[*Ans.* $6\frac{2}{3}$ lb. wt.

2. A small light ring can slide on a thin horizontal rail the co-efficient of friction between the ring and the rail being 0-6. A weight of 10 lb. is attached to the ring and a force P acts on the ring in the vertical plane containing the rail and in a direction inclined at an angle of 30° to the upward drawn

vertical. Find the frictional force when P=5 lb wt. and show that the ring will not move. Find also the least value of P which will cause the ring to slide. [*Ans.* $2\frac{1}{2}$ lb. wt., 5·88 lb. wt.

3. A plane is inclined to the horizontal at an angle α such that $\sin\alpha=\frac{3}{5}$, a particle of mass 10 gm. is gently placed on the plane. If the co-efficient of friction between the plane and the particle is 0·8 , prove that the particle can remain at rest, and calculate the frictional force exerted on the particle by the plane.

Calculate the least force acting along a line of greatest slope, which would cause the particle to move up the plane.

Ans. 6 gm. wt., 12·4 gm. wt.

4. A particle of weight 10 lb. is placed on a rough horizontal plane which is slowly tilted. If the particle begins to slip when the inclination of the plane to the horizontal is 60°, find the force of friction when the inclination of the plane is 45°.

Find also the least force, acting up a line of greatest slope, which will cause the partcle to begin to move up the plane when the inclination of the plane is 60°.

[*Ans.* $5\sqrt{2}$ lb. wt.. $10\sqrt{3}$ lb. wt.

5. A load of 30 lb. is held in equilibrium on a rough plane inclined at 45° to the horizontal by a horizontal force P in the same vertical plane as the line of greatest slope through the load. If the co-efficient of friction is $\frac{1}{2}$. find the value of P when the load is about to slip down the plane. [*Ans.* 10 lb. wt.

6. Find the least force which will move a weight of 80 lb. up a rough plane inclined to the horizontal at 30°, when the co-efficient of friction is $\frac{3}{4}$. [*Ans.* 73·6 lb. wt.

7. A body is supported on a rough inclind plane by a force acting along the plane ; supposing the greatest magnitude of the foree to be double the least magnitude, show that the inclination of the plane is given by $\tan\alpha=3\mu$.

8. If P_1 is the force which acting parallel to an inclined plane of angle α, is sufficient to draw a weight up and P_2 the force which will just let it be on the point of sliding down, show that

(*i*) $\frac{1}{2}(P_1+P_2)$ acting along the plane would support the same weight on a smooth plane of the same inclination.

and (*ii*) if $P_1=nP_2$, $\tan\alpha=\mu\dfrac{n+1}{n-1}$.

9. An elastic string of natural length l inches when used to pull a body of weight W lb. steadily along a rough level table, stretches $l/10$ inches. If the body is lifted vertically, the extension is $9l/20$ inches. Show that the co-efficient of friction of the table is 2/9. [P. U. 1940.)

10 A particle of weight w is placed on a rough plane inclined at an angle α (greater than the angle of friction) to the horizontal. It is attached to a string which lies along a line of the greatest slope, passes over a small smooth pulley at the top of the plane, and carries a light scale-pan hanging vertically at the end The particle is just prevented from slipping down the plane by a weight nw in the scale-pan. Show that the co-efficient of friction is given by $\mu=\tan\alpha-n\sec\alpha$.

Find the least weight that must be added to the scale-pan to drag the particle up the plane. [*Ans*, $2w(\sin\alpha-n$]

11. A weight can be just supported on a rough inclined-palne by a force P acting along the plane, or by a force Q acting horizontally, show that the weight is

$$PQ/\sqrt{Q^2\sec^2\phi-P^2}$$

where ϕ is the angle of friction.

12. Show that the difference between the greatest and the least force which, acting at an angle θ to the plane inclined at an angle α to the horizontal, sustains a weight W is

$$\frac{2W\cos(\alpha+\theta)\sin 2\lambda}{\cos 2\theta+\cos 2\lambda}$$

where λ is the angle of friction.

Miscellaneous Examples I.

1. *Forces P, Q act at a point O, and their resultant is R. If any transversal cut their lines of action in the points L, M, N respectively, show that*

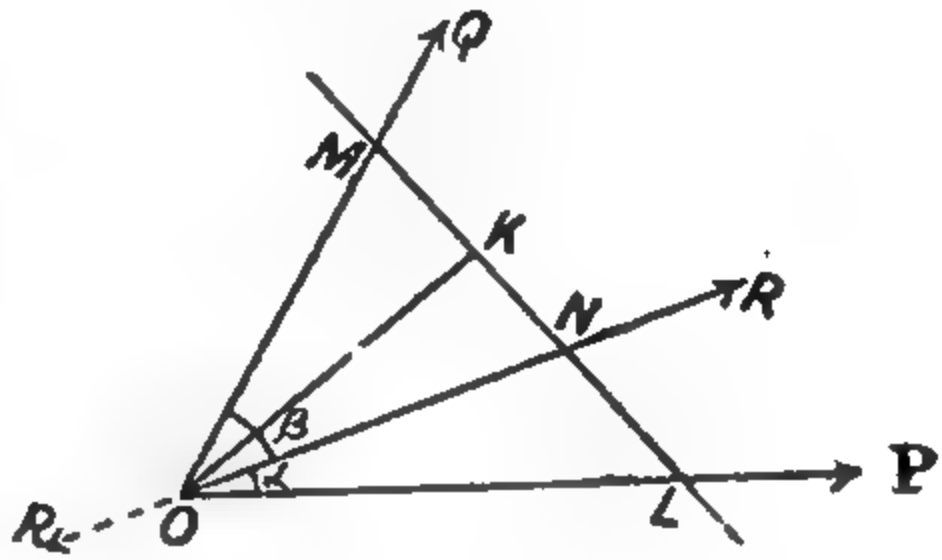
$$\frac{P}{OL}+\frac{Q}{OM}=\frac{R}{ON}$$

First Method. Draw OK perp. to LM. Resolving forces along OK, we have

$$P \cos \angle KOL + Q \cos \angle KOM = R \cos \angle KON$$

$$\text{i.e., } P\frac{OK}{OL} + Q.\frac{OK}{OM} = R.\frac{OK}{ON}$$

$$\therefore \quad \frac{P}{OL} + \frac{Q}{OM} = \frac{R}{ON}$$



Second Method. We have

$$P = \frac{P}{OL}.\ OL,\ Q = \frac{Q}{OM}.\ OM.$$

Now the resultant of forces $\frac{P}{OL}.\ OL$ acting along OL and $\frac{Q}{OM}.\ OM.$ acting along OM is $\left(\frac{P}{OL} + \frac{Q}{OM}\right)$ ON (Art. 3·1)

Hence $\left(\frac{P}{OL} + \frac{Q}{OM}\right).\ ON = R$ *i. e.*, $\frac{P}{OL} + \frac{Q}{OM} = \frac{R}{ON}$.

Third Method. If R be reversed, then the forces are in equilibrium

$\therefore$ by Lami's theorem, we have

$$\frac{P}{\sin \beta} = \frac{Q}{\sin \alpha} = \frac{R}{\sin (\alpha+\beta)} \qquad (i)$$

But since $2\triangle OLN + 2\triangle ONM = 2\triangle OLM$

$\therefore$ $OL.ON \sin \alpha + ON.OM \sin \beta = OL.\ OM \sin (\alpha+\beta)$

Equations (1) and (2) ; give the same result at once.

2. ABC is a triangle, with a right angle at A and AD is perpendicular on BC. Prove that the resultant of forces $\frac{1}{AB}$ acting along AB and $\frac{1}{AC}$ acting along AC is $\frac{1}{AD}$ acting along AD.

3. From a point P within a circle whose centre is fixed, straight lines PA_1, PA_2, PA_3, and PA_4 are drawn to meet the

circumference, all being equally inclined to the radius through P. Show that if these lines represent forces radiating from P, the resultant is independent of the magnitude of the radius of the circle. (P. U. 1934).

4. In a triangle OAB forces represented by OA. tan A and OB. tan B act along the sides OA and OB respectively. Show that the resultant is AB. tan A tan B and acts in the direction OD, the perpendicular from O on AB.

5. Forces represented by the sides AB, AC of a triangle ABC have a resultant whose line of action passes through the centre of the circumscribing circle. Prove that either AB=AC, or the angle BAC is a right angle.

6. Forces are represented by PA, PB, PC, AQ, BQ, CQ, where A, B, C, P, Q are points in a plane. Show that the resultant is represented in magnitude and direction by 3PQ and that it passes through the centroid of the triangle ABC.

7. If O be the centre of the circle circumscribing the triangle ABC and if forces act along OA, OB, and OC respectively proportional to BC, CA and AB, show that their resultant passes through the centre of the inscribed circle (P. U. 1937)

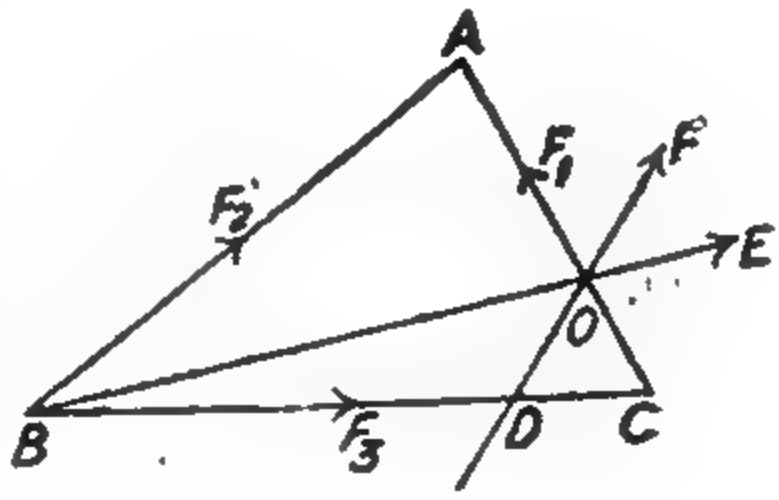
8. A very small ring of weight W is free to slide on a smooth circualr wire of radius a, fixed in a vertical plane. It is attached by a string of length l ($< a\sqrt{2}$) to a point of the wire at the same height as the centre. Show that the tension of the string is $\dfrac{(2a^2-l^2)W}{a\sqrt{4a^2-l^2}}$.

9. *Show that any force F can be resolved into three forces acting along the sides of a given triangle whose plane contains the line of action of F.*

The line of action of the force F must cut at least t o of the sides (or sides produced) of the given triangle ABC. Let it cut AC in O.

Join BO and produce it to E.

Now F can be resolved into two components along OA (*i.e.*, along CA, say F_1) and OE (*i.e.*, BO).



The force along BO can further be resolved into components (F_2, F_3) along BA and BC.

Thus F can be replaced by three components along the sides of the given triagle.

10. M is a point of trisection of the side AC of a triangle ABC which is nearer to A, and N is the point of trisection of the side AB which is nearer to B. Resolve a force represented in magnitude and direction by MN, into three forces acting each along a side of the triangle,

Since NA=2BN the forces MA and 2MB≡3MN, So that the force MN

≡$\frac{1}{3}$ MA and $\frac{2}{3}$ MB

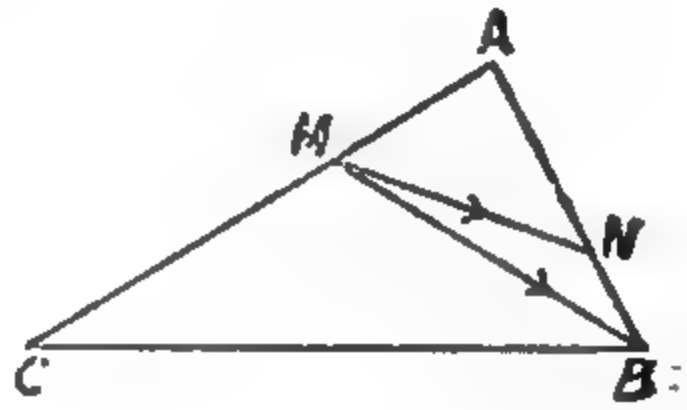
≡$\frac{1}{9}$CA and $\frac{2}{3}$ MB.

Again, since MC=2AM,

∴ forces CB and 2AB≡3MB,

so that $\frac{2}{3}$MB≡$\frac{1}{3}$ ⋋ 3MB≡

$\frac{2}{9}$CB and $\frac{4}{9}$ AB.



Hence the force MN≡$\frac{1}{9}$CA, $\frac{4}{9}$AB, and $\frac{2}{9}$CB.

11. Show that a force can be resolved into three components acting along three given lines, two of which are parallel, the lines and the forces being all coplanar.

12. *Show that any system of coplanar forces can be reduced to two forces acting through two given points in their plane.*

Take a point P on one of the forces, join it to the given points A and B. This force can be replaced by components along PA and PB. Other forces can similarly be dealt with, and then various forces acting at A and B can be combined into two forces through A and B.

13. Two forces given in magnitude act each through a fixed point and are inclined at a constant angle, show that their resultant also passes through a fixed point.

14. Three light elastic strings, AB, BC, CD each of natural length l and modulus λ. are fixed at A and D, two points in a horizontal line and at a distance $2l$ apart. Two particles each

of weight λ are attached at B and C respectively. Show that in the position of equilibrium AB and CD are inclined to the vertical at an angle θ given by $3 \tan \theta = 1 - 2 \sin \theta$.

15. A light string is fastened to two points A, D at the same level, the length of the string exceeding the distance AD, and particles of weight 2 lb and 1 lb., are fastened to it at two points B and C respectively. If AB, BC, CD make angles α, β, θ respectively with the horizontal, prove that

$$\tan \alpha = 2 \tan \theta \pm 3 \tan \beta.$$

16. A picture of breadth $2a$ is hung by two equal strings of length which are attached to the top corners of the picture, the other ends of the strings being attached to small light rings which can slide along a rough horizontal picture rail. If $2c$ is the greatest distance the rings can be apart without slipping, prove that the coefficient of friction is

$$\frac{c-a}{\sqrt{l^2-(c-a)^2}}.$$

17. A weight W rests on a rough horizontal plane and it is connected by a string over a smooth pully above the plane to weight w hanging vertically the sloping part of the string makes an angle θ with the horizontal. Prove that there will be no slipping, if $w \cos(\theta - \lambda) < W \sin \lambda$, where λ is the angle of friction between W and the plane.

18. Two weights P, Q resting on a double inclined plane, are connected by a fine string over a common vertex and Q is on the point of motion down the plane. Prove that the greatest weight which can be added to P without disturbing equilibrium is

$$\frac{Q \sin 2\lambda \sin(\alpha+\beta)}{\sin^2\alpha - \sin^2\lambda}$$

α, β being the angles of inclination of the planes and λ the angle of friction. (*D. U. Hons.* 1925)

REVISION QUESTIONS. I

1. State Hooke's Law for Elastic Strings.

A heavy particle is hung from two pegs, which are in a horizontal line and distant $2c$ apart by light elastic strings of natural lengths a, b and moduli λ, μ respectively. In the posi-

tion of equilibrium the strings are inclined at angle a to the vertical. Show that $\sin a = c(\lambda b - \mu a)/ab(\lambda - \mu)$.

2 Two elastic strings of the same natural length are such that a force of 1 lb stretches the first 1 inch and the second 3 inches. (*i*) The strings are fixed, each at one end, to the same point and the other ends are attached to a weight of 4 lb., so as to support it in equilibrium. Find the tensions in the two strings. (*ii* The weight of 4 lb, is suported by two strings, the first being suspended from A and the second from B vertically above A. If the tension in each string is the same, find the distance AB. [*Ans* 3 lb ; 1 lb , 4″.]

3. Prove that the components of a force F in two directions making angles θ and ϕ with its line of action (on opposite sides of it) are

$$F \sin\theta/\sin(\theta+\phi) \quad \text{and} \quad F \sin\phi/\sin(\theta+\phi)$$

ABC is an equilateral triangle of side a. The line of action of F cuts AB at a point between A and B and distant $\frac{2a}{3}$ from A, and AC at a point between A and C and distant $\frac{a}{3}$ from A. Resolve it graphically or otherwise into three forces acting along the sides of ABC. [D.U. 1942].

[*Ans.* $\frac{2\sqrt{3}}{9}F$, $\frac{\sqrt{3}}{9}F$, $\frac{4\sqrt{3}}{9}F$.

4. Two forces P, Q act at a point along straight lines making an angle a with each other, and have a resultant R, two other forces P′, Q′ acting along the same two lines have a resultaut R′. Show that the angle between the lines of action of these resultants is

$$\cos^{-1}[PP' + QQ' + (PQ' + P'Q)\cos a]/RR'.$$

5. In Q. 4, show that the directions of R and R′ will also include an angle a, if

$$PP' + QQ' + 2PQ' \cos a = 0 \text{ or } PP' + QQ' + 2P'Q \cos a = 0.$$

6. Two forces act along the sides CA, CB of a triangle ABC, their magnitudes being proportional to cos A and cos B.

Prove that their resultant is proportional to sin C, and its direction divides the angle C into two portions

$$\tfrac{1}{2}(C+B-A),\ \tfrac{1}{2}(C+A-B)$$ (P. U. 1924)

7. Prove that the resultant of two forces, acting at a point O in directions OA and OB, represented in magnitude by l.OA and m.OB, is reprerented by $(l+m)$ OC where C is a point in AB such that l.CA$=m$. CB.

ABCD.....is a polygon of n sides and forces act at a point O parallel and proportional to AB, 2BC, 3CD etc. Show that their resultant is parallel and proportional to $(n-1)$ OA, where O is the centroid of all the points B, C, D...excluding A.

8. A weight is supported on a smooth plane of inclination α by a string inclined to the horizon at angle γ. If the slope of the plane be increased to β. and the slope of the string remains unaltered, the tension of the string is doubled. Find the relation between α, β and γ. (D. U. 1933.)

[*Ans* $2\sin\alpha\cos(\gamma-\beta)=\sin\beta\cos(\gamma-\alpha)$.

9. A body of weight W rests on a rough inclined plane whose inclination α to the horizontal is greater than β, the angle of friction. It is held at rest by an elastic string of natural length l attached to it and to a point on the plane. If λ is the modulus of elasticity, show that the string is lengthened by an amount equal to

$$Wl\sin(\alpha-\beta)/\lambda\cos\beta.$$

10. Weights P and Q of two different substances, coefficients of friction μ and μ', are supported on a rough double inclined plane of angles α and β by means of a string passing over the vertex. If the weight P be on the point of descending, prove that

$$\frac{P}{Q}=\frac{\sin\beta+\mu'\cos\beta}{\sin\alpha-\mu\cos\alpha}$$

11. In Q 10, if $\mu=\mu'=\tan\lambda$ and $\beta=90-\alpha$, show that $\frac{Q}{P}$ must lie between $\tan(\alpha-\lambda)$ and $\tan(\alpha+\lambda)$.

12. *Forces proportional to the sides of a closed polygon act along lines which bisect at right angles those sides. If in going round the polygon, the forces act all towards the right or towards the left, show that they form a system in equilibrium.*

CHAPTER IV

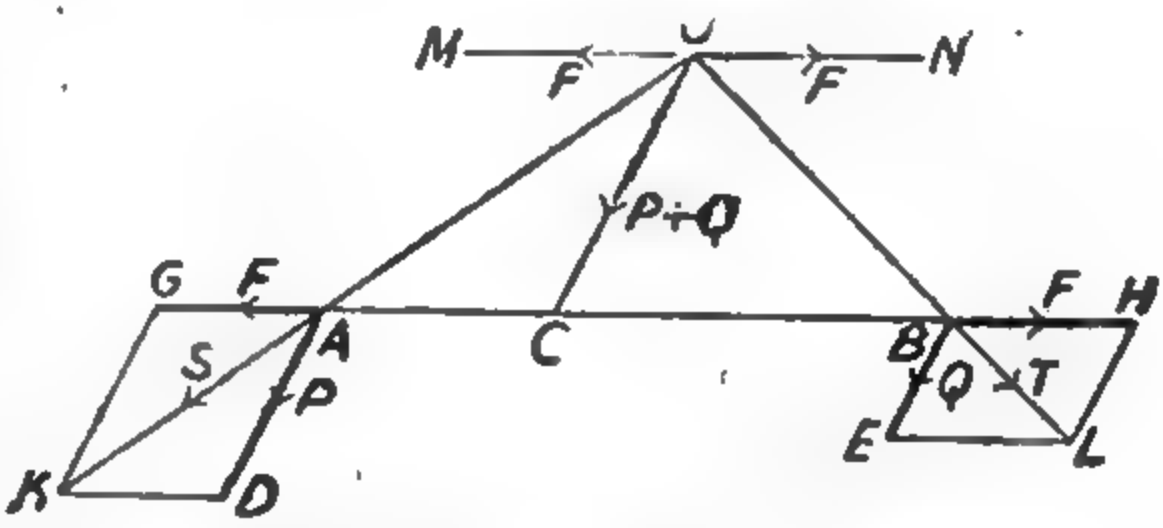
PARALLEL FORCES—MOMENT.

4.1. In the preceding chapter we have shown how to deal with forces acting at a point, basing the discussion on the law of parallelogram of forces. We shall now consider the composition of parallel forces based on the same fundamental law.

4.11. Resultant of two parallel forces.

Case I. *Like parallel forces, i.e.*, forces acting in the same parallel direction.

Let P and Q be two like parallel forces acting at points A and B of a rigid body. Let them be represented by AD and BE, respectively.



At A and B introduce two forces each equal to F, in opposite directions, acting in the line AB, represented by AG and BH. These forces, being equal and opposite, balance and they do not affect the resultant of P and Q.

Let S be the resultant of P and F acting at A and T, the resultan of Q and F acting at B. Let the lines of action of S and T meet in O and let their points of action be shifted to O.

Draw OC parallel to the direction of P or Q meeting AB in C and through O draw MN parallel to AB.

At O resolve S into components in their original directions, P acting along OC and F acting along OM. Similarly resolve T into the components Q along OC, and F along ON.

The two forces, each equal to F, acting along OM and ON balance and we are left with the force P+Q acting along OC.

Hence the resultant of P and Q is P+Q acting along OC, *i.e.* acting at C parallel to P or Q.

Now, we find the *position of C*, where the resultant cuts the line AB.

From the similar triangles AOC, AGK, we have

$$\frac{AC}{OC}=\frac{AG}{GK}=\frac{F}{P} \quad \text{or} \quad P.AC=F.OC$$

Similarly from triangles OCB, BHL, we have

$$\frac{BC}{OC}=\frac{BH}{HL}=\frac{F}{Q} \quad \text{or} \quad Q.BC=F.OC.$$

$$\therefore \quad P.AC=Q.BC \quad \text{or} \quad P/BC=Q/AC.$$

Cor. *If a force R balance the two like parallel forces P and Q, it must be equal to P+Q and must act through C in the direction CO (in the plane of P and Q, parallel to them), such that.*

$$\frac{P}{CB}=\frac{Q}{AC}=\frac{P+Q}{CB+AC}=\frac{R}{AB} \qquad \ldots\ldots(1)$$

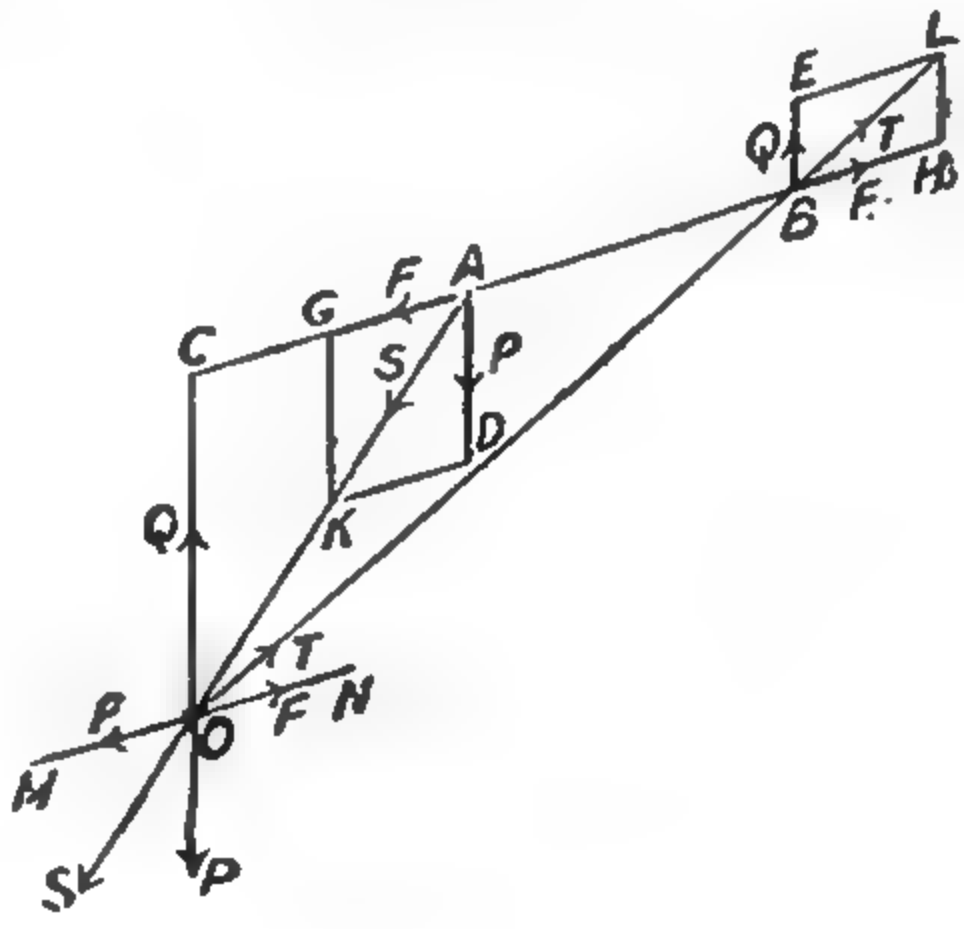
Case II. *Unlike parallel forces, i.e.,* forces acting in opposite parallel directions.

Let P and Q (P>Q) be the two unlike parallel forces acting at points A and B of a body and let them be represented by AD BE.

At A and B introduce two forces, F, in opposite directions, acting in the line AB and represented by AG and BH.

The forces F balance and do not affect the action of P and Q.

S, the resultant of P and



F at A is represented by AK and T, the resultant of Q and F at B is represented by BL.

Produce AK and LB to meet in O (they will always meet unless they are parallel which happens when P and Q are equal). The forces S and T may be supposed to act at O.

Draw OC parallel to line of action of P or Q meeting BA produced in C, and through O draw MN parallel to AB.

At O resolve S into components in their original direction P acting along CO and F along OM. Similarly resolve T into components Q along OC and F along ON.

The forces F along OM and ON balance and we are left with forces P along CO and Q along OC *i.e.*, P—Q acting alo g CO.

Hence the resultant of P and Q is P—Q along CO *i.e.*, acting through C parallel to P or Q in the sense of the greater force P.

Now, to find the position of the point C, we have from similar triangles AOC, AGK, AC/OC=AG/KG=F/P

i.e., P.AC=F.OC.

Similarly from triangles BOC, BHL, Q.BC=F.OC

$\therefore$ P.AC=Q.BC *i.e.*, P/BC=Q/AC

Cor. *If a force R balances, the two unlike parallel forces P and Q, it must be equal to P—Q and must pass through C in the direction OC (in the plane of P and Q, parallel to them) such that*

$$\frac{P}{BC} = \frac{Q}{AC} = \frac{P-Q}{BC-AC} = \frac{R}{AB} \qquad \ldots\ldots(2)$$

From (1) and (2), it follows that :—

If three parallel forces are in equilibrium, each is proportional to the distance between the other two.

This is analogous to Lami's Theorem.

4.12. It is important to note that *the position of the point C is independent of the direction of the parallel forces.*

Also that the line of action of the resultant of two forces lies in the plane determined by the lines of action of the forces.

4.13. In case II if the forces P and Q are equal in magnitude and opposite in direction, the lines of action of the forces S and T become parallel and they do not meet. In this case the forces P and Q are not equivalent to a single resultant. Such a system of two equal and unlike parallel forces is called **a couple**, the properties of which will be considered in the next chapter.

EXAMPLES XI

1. The resultant of two like parallel forces P, Q passes through a point O, when P is increased by R and Q by S, the resultant still passes through O, and also when Q, R replace P, Q respectively prove that

$$R=Q^2/P \qquad \text{and} \qquad S=Q^3/P^2.$$

Let x and y be the distances from O of the lines of action of P and Q, then by the question, we have

$$Px=Qy \qquad \ldots\ldots(1)$$

and
$$(P+R)\,x=(Q+S)\,y \qquad \ldots\ldots(2)$$

Also
$$Qx=Ry \qquad \ldots\ldots(3)$$

Eliminating x and y from (1) and (3), we get

$$PR=Q^2 \qquad \text{or} \qquad R=Q^2/P \qquad \ldots\ldots(4)$$

Also eliminating x and y from (1) and (2), we have

$$P\,(Q+S)=Q\,(P+R) \qquad \text{or} \qquad PS=QR$$

$$\therefore \quad S=QR/P=Q^3/P^2 \qquad [\text{by (4)}]$$

2. The resultant of two like parallel froces is 12 lb. and it acts at a distance of 2 in. from the larger component which is 8 lb. ; find its distance from the smaller component.

(Ans. 4 in.)

3. Three points A, B, C are in straight line and AB=BC=10 in. Like parallel forces of 7 lb. wt. and 3 lb. wt. act at A and B respectively. Find the magnitude and line of action of the resultant of these forces.

Find at what point in AC, produced if necessary, a third parallel force of 5 lb. wt. acting in opposite sense, should be introduced so that the resultant of these forces may act through C.

(Ans. 10 lb. wt. 3 in from A : 14 in from A and 34 in from C.)

4. A man carries a bundle at the end of a stick which is placed over his shoulder, show that the pressure on his shoulder varies inversely as the distance between his hand and the shoulder.

5. P and Q are like parallel forces. If P is moved parallel to itself through a distance d, prove that the resultant of P and Q moves through a distance $Pd/(P+Q)$,

6. A uniform rod of length $2l$ and weight W, is lying across two pegs on the same level d feet apart. If neither peg can stand a stress greater than T, show that the length of the rod which can project beyond either peg cannot be greater than

$$l-\frac{d(W-T)}{W}.$$

7. The lines of action of two forces P and Q and their resultant R are cut by another line in the points A, B and C respectively. P and Q are each resolved into two forces, one parallel to AB and the other parallel to R. Prove that the components parallel to R are to each other as BC : AC

8. P and Q are two unlike parallel forces, when P is doubled it is found that the line of action of Q is midway between the lines of action of the new resultant and the original resultant. Find the ratio of P and Q. (Ans $P=\frac{3}{4}Q$)

9. Three equal like parallel forces act at the middle points of the sides of a triangle, show that their resultant passes through the intersection of the medians of the triangle.

10. P and Q are two like parallel forces acting at A and B. Show that if they interchange position, the point of application of the resultant is displaced a distance $(P-Q)AB/(P+Q)$.

11. ABC is a triangle and O a point in the plane of the triangle. A force R acts along AO. Resolve R into two forces parallel to it and acting at B and C respectively, when

(a) O is the orthocentre of $\triangle$ ABC

(b) O is the circumcentre of $\triangle$ ABC. (P. U. 1942)

Ans. (a) $\frac{Rb}{a}\cos C, \frac{Rc}{a}\cos B$. ($b$) $\frac{Rb\cos B}{a\cos(B-C)}, \frac{Rc\cos C}{a\cos(B-C)}$.

4.2. Centre of a number of parallel forces. Let a number of parallel forces act at fixed points of a body. If the forces are not all in the same sense, let us take them in two groups, each group consisting of like parallel forces, *i.e.*, forces in the same sense, so that the direction of any force belonging to one group is opposite to the direction of a force of the second group.

Let the forces P_1, P_2, P_3,.....acting at A_1, A_2, A_3,..... belong to one group and Q_1, Q_2, Q_3,..... acting at B_1, B_2, B_3,... belong to the other.

Now consider the forces of the first group. By Art. 4.11, the resultant of P_1 and P_2 is P_1+P_2 acting at G_1 on A_1A_2 such that $P_1.A_1G_1=P_2.G_1A_2$. Again the resultant of P_1+P_2 at G_1 and P_3 at A_3 is $P_1+P_2+P_3$ at G_2 on G_1A_3 such that $(P_1+P_2).G_1G_2=P_3.G_2A_3$ and so on. In this manner, we arrive at the resultant $P_1+P_2+P_3+\ldots=R_p$ acting at a point G_p. Similarly by combining the forces of the second group we will arrive at the resultant $Q_1+Q_2+Q_3+\ldots=R_q$ acting at a point G_q.

Suppose these two resultants are *unequal* in magnitude, then on combining them by Case II, Art. 4.11, we arrive at the final resultant of all the forces acting at a point G such that

$$R_p \cdot G_p G = R_q G_q G.$$

This point G is called the **centre of parallel forces.** Its position depends only on the positions of the points of application of the forces and their relative magnitude. It is *independent of the direction* of the forces, so that if the forces are turned about their points of application through the same angle so as to remain parallel, the position of G remains the same.

In case R_p and R_q are *equal and collinear* in action, the resultant vanishes and the system is in equilibrium.

Again if R_p and R_q are equal in magnitude, but not collinear in action, the system reduces to a *couple* and is not equivalent to a single resultant.

Notice that in the above discussion the parallel forces are not restricted to lie in one plane.

4.21. A very important application of the above article occurs in connection with the weight of a body. A body may

be considered as made up of a very large number of particles and if the size of the body is small compared with that of the earth, tho forces on all the particles of it will be very nearly parallel. Thus the weights of the particles form a system of parallel forces and their centre G is known as the *centre of gravity* (C.G.) of the body.

The position of the centre of gravity of certain bodies is obvious by their symmetrical nature, for example :—

(*i*) C.G. of a uniform straight rod is its middle point.

(*ii*) C.G. of a uniforn plate in the form of a parallelogram or a rectangle or a square, and a cube is the centre (*i.e.*, the point of intersection of the diagonals).

(*iii*) C.G. of a uniform circular disc, or a circular ring, or a sphere is the centre.

(*iv*) C.G. of a uniform right circular cylinder is the middle point of its axis.

The position of C.G. for various other bodies will be considered in Chapter V.

As a knowledge of the C.G. of certain well known bodies is necessary in the solution of a large number of problems, we quote the following results for reference :—

(*v*) C.G. of a uniform triangular lamina is the point of intersection of its medians.

(*vi*) C.G. of a uniform right circular cone lies on its axis at a distance $\frac{3}{4}$th of the height from the vertex.

(*vii*) C.G. of a uniform solid hemisphere is on the line drawn through the centre perpendicular to the plane face at a distance $\frac{3}{8}$th of its radius from the centre.

4.3. Theorem II B. *The sum of the resolved parts of a pair of parallel forces, not forming a couple, along any line in their plane is equal to the resolved part of their resultant along the same line.*

Let P and Q be the two parallel forces and R, their resultant. Let the forces be inclined at an angle θ to the line OX, along which the forces are to be resolved. Then R will also be inclined at the same angle with OX.

Now, since $R=P+Q$ or $P-Q$ according as the forces are like or unlike.

$\therefore$ $R\cos\theta=P\cos\theta\pm Q\cos\theta$

i.e., the resolved part of the resultant R along OX

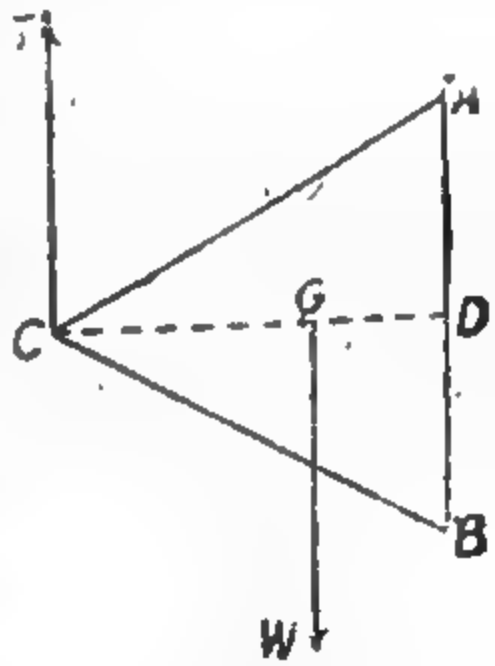
=sum of the resolved parts of the component forces.

Note The proof is obviously applicable to a number of parallel forces.

431. Equilibrium of three parallel forces. *If a rigid body is in equilibrium under the action of three forces, two of which are parallel, the third force must also be parallel to the other two and must lie in their plane.*

This is so because the third force must be equal and opposite to the resultant of the other two.

Example. *A uniform equilateral triangular lamina ABC of weight W has the vertex A hinged to a fixed point about which it can turn freely in a vertical plane and rests with AB vertical by a vertical string attached to C. Find the tension in the string and the magnitude and direction of the reaction R at the hinge.*



Since two of the forces, the tension T acting at C and the weight W acting at G, the C. G. of the lamina are vertical, therefore the third force, the reaction R at A must also be vertical.

$\therefore$ $T+R=W$,

and $T.CG=R.GD$ *i.e.* $2T=R$ ($\because CG=2GD$)

$\therefore$ $3T=W$ *i. e.* $T=\frac{1}{3}W$ and $R=2T=\frac{2}{3}W$.

EXAMPLES XII.

1. A uniform beam AB of length 4 ft. is supended from two points C, D at the same level by strings AC, BD. If the distance between C and D is 2 ft, and the string AC is 1 ft. long, find how long the string BD must be in order that AC may be vertical when the system is in equilibrium.

[Ans. $1+2\sqrt{3}=4{\cdot}46$ ft.

2. A uniform circular ring of weight W, free to move in a vertical plane rests over a fixed peg at A and is suspended by a vertical string attached to a point B of the ring. If AB subtends an angle of 135° at the centre and in the position of equilibrium, the diameter through B is horizontal, find the tension in the string. [Ans. $W(\sqrt{2}-1)$

3. A right circular cone is suspended by two light vertical strings of equal length one attached to the vertex of the cone, the other to a point on the rim of its base. The points of suspension are at the same horizontal level and the tensions in the strings are equal. Find the semi-angle of the cone.

Ans : $\cos^{-1}\sqrt{\frac{2}{3}}$

4. An equilateral triangular lamina weighs 10 lbs. A weight of 10 lb. is hung by a cord from an angle, and the whole is hung by a second cord attached to another angle. Find the tangent of the angle which the line joining these angular points makes with the verticai. Ans. $\sqrt{3}/9$

5. A lamina in the form of an isosceles triangle ABC, in which the angle C is a right angle, can turn freely in a vertical plane about A which is fixed. The lamina is held in equilibrium by a vertical string attached at B, the vertex C being downwards and B below the level of A. Prove that the tension in the string is $\frac{1}{2}W\ (1-\frac{1}{3}\tan\theta)$ where W is the weight of the lamina and θ the angle which AB makes with the horizontal.

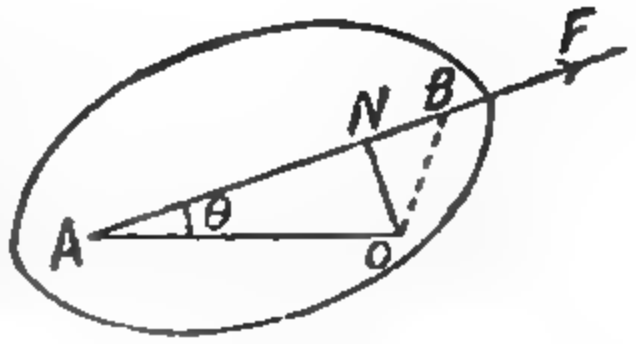
6. A uniform bar AB, of weight W and length l. is smoothly hinged at its end A, which is at a distance $2l$ vertically below a smooth horizontal wire. The bar can turn in the vertical plane through the wire, and the other end B of the bar is attached by a light elastic string of natural length l to a smooth ring, which slides on the wire. Find the distances of B from the wire in the positions of equilibrium, given that the tension in the string is W when its length is double its natural length. [Ans. l, $\frac{3}{2}l$, $3l$.

MOMENT

4·4. Moment *of a force about a point is the product of the force and the perpendicular distanee of the point from the line of action of the force*

Suppose a point O of a rigid body (*e. g.*, a disc) is held

fixed while a force F acts on it. If F does not pass through O, it has a tendency to rotate the body about O. This tendency of rotation depends not only on F, the magnitude of the force, but also on p, the distance of O from its line of action and is measured by the product Fp which is the moment of F about O.



4·41. Let A be any point on the line of action of F, OA $=r$ and $\angle OAN=\theta$,

then $F.p=F.\ r\sin\theta=r.F\sin\theta=AO\times$ Resolved part of $F\perp OA$.

4·42. Geometrical Representation of Moment. If AB represents F, on some scale, then $Fp=AB.p=$twice the area of $\triangle OAB$, so that *the measure of the moment of a force about any point is geometrically equivalent to the measure of twice the area of the triangle formed by joining the point to the extremities of the line representing the force.*

4·43. Sign of Moment. The direction of rotation of a

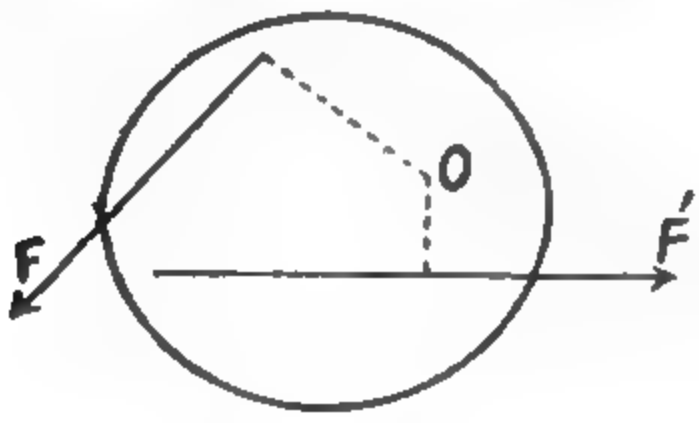


Fig. 1.

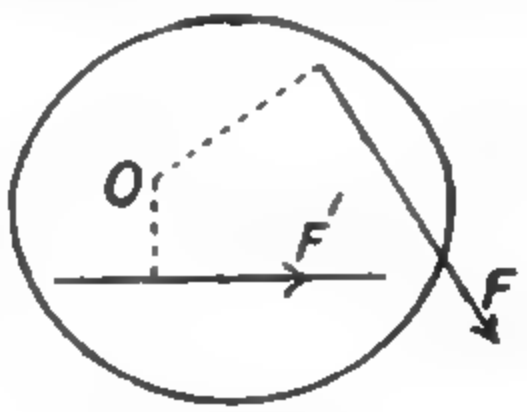
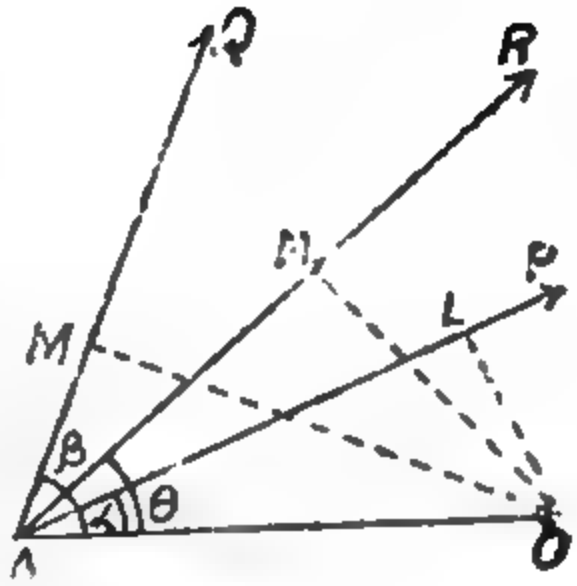


Fig. 2.

body about a point due to the action of a force depends on the relative position of the point and the line of action of the force. Thus in Fig. 1, both F and F′ tend to rotate the body about O in the same (counter-clockwise) sense and we assign the same sign (usually taken positive) to their moments. But in Fig. 2, F and F′ tend to cause rotation about O in opposite senses and opposite signs are assigned to their moments.

When dealing with the moment of a number of forces about a point, the signs of the moments must not be ignored.

4·5. Theorem III. Varignon's Theorem on Moments. *The sum of the moments of two forces (not forming a couple), about a point in their plane is equal to the moment of their resultant about the same point.*



Case I. *When the forces act at a point.*

Let P and Q be the forces acting at the point A, R their resultant and O the point about which moments are to be taken. Let $\angle OAP=\alpha$, $\angle OAQ=\beta$ and $\angle OAR=\theta$, then

the sum of the moments of P and Q about O

$=P.OL+Q.OM$

$=P\ OA \sin\alpha+Q.OA \sin\beta$

$=OA\ (P \sin\alpha+Q \sin\beta)$

$=OA\times$sum of the resolved parts of P and Q perp. OA.

$=OA\times$resolved part of R perp. OA. (by Art. 2·4)

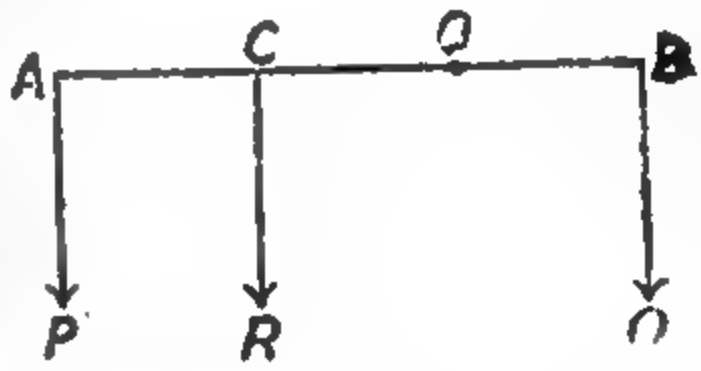
$=OA.R \sin\theta$ $R.OA \sin\theta=R.ON=$moment of R about O.

Note—The proof is applicable to a number of co-planar and concurrent forces.

Case II. *When the forces are parallel.*

Let P and Q be two parallel forces and R, their resultant.

From O draw a line perpendicular to the lines of action of P and Q, meeting them in A and B and meeting R in C.



Then $P+Q=R$ and $P.AC=Q.CB$ .....(1)

The sum of the moments of P and Q about O

$=P.AO-Q.OB=P.(AC+CO)-Q.(CB-CO)$

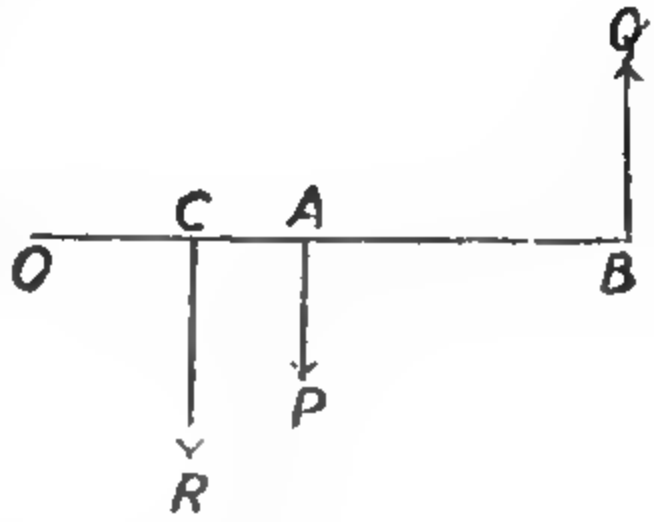
$=P.AC-Q.CB+(P+Q)CO.$

$=R.CO$ from (1)

=the moment of R about O.

This proof is applicable to all the positions of the point O, the moment of the forces being taken with their proper signs.

For example, let the forces P and Q be unlike (P>Q) and let O be outside these forces.



From O, draw OCAB perpendicular to the lines of action of the forces.

In this case

R=P−Q and P.CA=Q.CB.

The sum of the moments of P and Q about O.

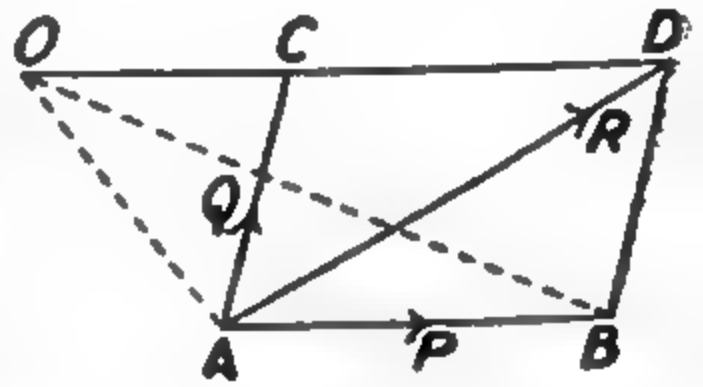
=Q.OB−P.OA=Q (OC+CB)−P.(OC+CA)

=(Q−P).OC+Q CB−P.CA= −(P−Q).OC= −R.OC

=the moment of R about O.

Cor. *The sum of the moments of the forces about any point lying on the line of action of their resultant vanishes.*

451. The following proof of Case. I is based on the geometrical representation of a moment.



Let the forces P and Q act along the lines AB and AC. Through O draw OC parallel to the direction of P meeting the line of action of Q in C.

Let AC *now* represent the force Q on some scale and on the same scale let AB represent P. Complete the parallelogram ABCD, so that AD represents the resultant R, then

the sum of the moments of Q and P about O

=2△OAC+2△OAB=2△OAC+2△ABD

=2△OAC+2△CAD

=2△OAD=moment of R about O.

The proof is applicable to all positions of the point O, the moments of the forces being taken with their proper signs.

(The student is advised to prove the proposition when O lies within the argle CAD).

EXAMPLES XIII

1. Forces 3, 4, 5 lb. wt. act along the sides BC, CA, AB of an equilateral triangle ABC of side 2 ft. in the directions indicated by the order of the letters. Find the moment of this system of forces about (*i*) A (*ii*) B (*iii*) C (*iv*) the C. G. of the triangle. [*Ans.* $3\sqrt{3}$, $4\sqrt{3}$, $5\sqrt{3}$; $4\sqrt{3}$ lb. ft.]

2. Forces 3, 5, 3 and 5 lb. wt. act along the sides AB, CB, DC, DA respectively of a square of sides 4 ft. Find the sum of their moments (*i*) about B (*ii*) about C (*iii*) about the centre of the square [*Ans.* 8, 32, zero lb. ft.]

3. Forces of 1, 2, 3, 4, 5 and 6 lb. wt. act respectively along the sides AB, BC, CD, DE, EF, FA of a regular hexagon whose side is 4 ft., find the sum of the moments of the forces (*i*) about the centre of the hexagon (*ii*) about A.
[*Ans.* $42\sqrt{3}$, $42\sqrt{3}$ lb. ft.]

4. ABCD is a square of side 10 ft. A certain force P passes through A and has moments of 20 and 30 lb. ft. in opposite sense about B and D respectively.

Find the angle which the line of action of P makes with AB. Also find the magnitude of P and its moment about C.
[*Ans.* $\tan^{-1}\frac{2}{3}$, $\sqrt{13}$ lb. wt. ; 10 lb. ft.]

4.6. Resultant of a number of Coplanar forces acting on a rigid body. A system of coplanar forces which are concurrent were treated in the last chapter. In what follows, we shall deal with a system of coplanar forces which are not necessarily concurrent.

4.61. Theorem IV. *Any system of forces acting in the same plane upon a rigid body can be reduced to either a single force or a single couple as their resultant.*

Any *three* forces of the system can always be reduced to *two* forces, for any one of the three can be compounded with at least one of the remaining forces unless it forms a couple with both of them, in which case, the latter two being equal and like parallel forces, can themselves be compounded into a single force.

For example, if P_1, P_2 P_3 be any three forces of the system,

GOVT. C... SRINAGA... ...GE

and if P_1 does not form a couple with P_2 then either by the law of parallelogram of forces (if they meet) or by the law of parallel forces (if they are parallel), the two can be combined and reduced to one force. In case P_1 forms a couple with P_2 but does not form a couple with p_3, it can be combined with P_3, and the two are thus reduced to a single force.

If however, P_1 forms a couple with P_2 as well as with P_3 it means that P_2 and P_3 are equal like parallel forces reducible to a single force as their resultant.

Thus it is always possible to reduce any *three* forces to *two*.

Now by taking another force of the system, with the two forces obtained before, we can again reduce these forces to two and so on.

In the end we are left with two forces, which either form a couple or are equivalent to a single force.

4·62. Theorem V. Generalised theorem of Moments. If a system of coplanar forces acting on a rigid body reduce to a single force as its resultant, then

the algebraic sum of the moments of the forces, about any in their plane, is equal to the moment of their resultant about the same point.

Let O be the point about which moments are to be taken.

Take two forces of the system not forming a couple, let them be P_1, P_2 and let R_1 be their resultant, then by Art. 4.5, we have

Moment of R_1 about O = sum of the moments of P_1 and P_2, about O,

Now combine R_1 with some other force, say P_3 not forming a couple with it and obtain the resultant R_2 then

Moment of R_2 about O = sum of the moments of R_1 and P_3
= sum of the moments of P_1, P_2, and P_3.

Proceeding in this manner, avoiding two forces forming a couple and repeating at every step, the process of taking moments of two forces, we arrive at the final resultant R of the system and thus we get

moment of R about O = the algebraic sum of the moments of P_1, P_2.....P_n.

Note. If at any stage, any of the resultants R_1, R_2, R_3... say R_p, form a couple with each of the remaining forces it will mean that the rest of the forces are all equal and like parallel forces. In such a case, leave R_p for a time and bombine any two of these like parallel forces and then proceed on as before.

4.621. From the above theorem it follows that :—

If a number of coplanar forces are in equilibrium, the algebraic sum of their moments about a point in their plane vanishes.

Also if a number of coplanar forces have a eingle force as their resultant, the sum of their moments about any paint on the line of action of the resultant vanishes and conversely if sum of their moments about a point in their plane vanishes, the resultant must pass through the point.

This fact is usually employed in problems requiring the line of action of the resultant of a number of coplanar forces acting on a rigid body.

4.63. Kheorem VI. Generalised theorem of resolved parts. If a system of coplanar forces acting on a rigid body reduce to a single force as its resultant then,

the algebraic sum of the resolved parts of the forces along any line in their plane, is equal to the resolved part of their resultant along the same line.

Let OX be the given line along which the forces are to be resolved.

Take two forces of the system not forming a couple, let them be P_1, and P_2 and let R_1 be their re-ultant, then whether P_1, P_2 are parallel or they meet, by Arts. 4.3 and 2.4, we have

resolved part of R_1 along OX=sum of the resolved parts of P_1 and P_2.

Now combine R_1 with some other force, say P_3 not forming a couple with it, and obtain the resultant R_2, then

resolved part R_2 along OX

=sum of the resolved parts of R_1 and P_3.

=sum of the resolved parts of P_1, P_2 and P_3.

Proceeding in this manner and repeating the argument

about the sum of the resolved parts of two forces, we arrive at the final resultant R of the system and thus we get,

resolved part of R along OX = the algebraic sum of the resolved parts of P_1, P_2.....P_n

Note —The note of Art. 4.62 applies to this theorem also.

4 631. Thus, if the line of action of the forces P_1, P_2, P_3.....make angles α, β, γ.....and the line of action of their resultant R makes θ with OX, then

$$R\cos\theta = P_1\cos\alpha + P_2\cos\beta + P_3\cos\gamma + \ldots\ldots = X \quad \ldots(1)$$

Similarly, resolving along the line OY perpendicular to OX, we have

$$R\sin\theta = P_1\sin\alpha + P_2\sin\beta + P_3\sin\gamma + \ldots\ldots = Y \quad \ldots(2)$$

Equations (1) and (2) give

$$R^2 = X^2 + Y^2 \qquad \text{and} \qquad \tan\theta = Y/X.$$

These results give the magnitude and direction of the resultant force, when it exists.

4 632. If R=0, then X and Y both vanish, so that it follow that :—

If a system of coplanar forces does not reduce to a force the algebraic sum of their resolved parts along a line in their plane vanishes and it also vanishes along a line perpendicular to the former.

Also, if the system reduces to a force and if X=0 *i.e.*, the sum of the resolved parts along a given line in their plane vanishes, the resultant force acts along a line perpendicular to the given line.

Also, if X=0 as well as Y=0 *i.e.*, the resolved parts along two perpendicular straight lines in the plane of the forces separately vanish, the resultant is not a force, it may be couple (Art. 4.61) or the forces may be in equilibrium

4.64. Example ; (*i*) *Forces X, Y, Z act along the sides BC, CA, AB of triangle ABC. Show that the resultant of the forces passes through the circumcentre and the incentre of the triangle if*

$$X : Y : Z = \cos B - \cos C \; ; \; \cos C - \cos A : \cos A - \cos B.$$

Let R be the circumradius and *r* the inradius of the

triangle, then the distances of the sides BC, CA, AB from the circumcentre O are R cos A, R cos B and R cos C, and from the incentre, I, each distance is equal to r.

Since the resultant passes through the points O and I, the sum of the moments of the forces about these points must separately vanish.

Hence $X.R\cos A+Y.R\cos B+Z.R\cos C=0$

or $X\cos A+Y\cos B+Z\cos C=0$ (1)

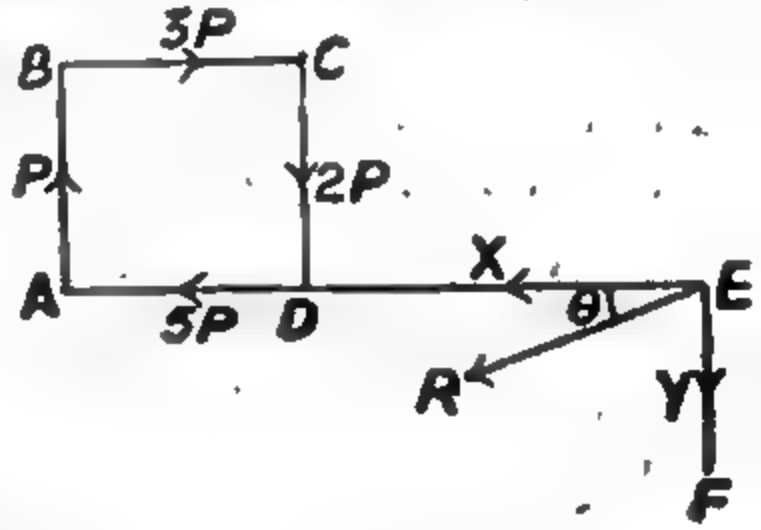
and $X.r+Y.r+Z.r=0$ or $X+Y+Z=0$ (2)

$\therefore$ from (1) and (2), we have

$$\frac{X}{\cos B-\cos C}=\frac{Y}{\cos C-\cos A}=\frac{Z}{\cos A-\cos B}$$

Example (ii). *Forces P, 3P, 2P, 5P act along the sides AB, BC, CD, DA of a square ABCD. Find the magnitude and direction of their resultant and prove that it meets AD produced at a point E such that AE : DE=5 : 4.* (*D.U. 1940.*)

Let the line of action of the resultant meet AD produced at E and let AD=a, DE=x, then since the sum of the moments about E must vanish, we have



$$2Px-P(a+x)-3Pa=0$$

i.e., $x=4a$.

$\therefore$ DE : AE=$4a$: $5a$=4 : 5.

Now let X, Y be the components of the resultant R along ED and EF and θ the angle its line of action makes with ED. By resolving forces along these lines, we have

$X=5P-3P=2P$ and $Y=2P-P=P$.

$$\therefore\ R=\sqrt{X^2+Y^2}=\sqrt{5}P \text{ and } \tan\theta=\frac{Y}{X}=\frac{P}{2P}=\frac{1}{2}.$$

EXAMPLES XIV.

1. Three forces P, Q, R act along the sides BC, CA, AB of a Δ ABC. Show that their resultant passes through

(i) the centroid if P cosec A+Q cosec B+R cosec C=0

(*ii*) the orthocentre if $P \sec A + Q \sec B + R \sec C = 0$

(*iii*) the orthocentre and centroid if

$$P : Q : R = \sin 2A \sin (B-C) : \sin 2B \sin (C-A) : \sin 2C \sin (A-B)$$

(*Agra 1940*)

(*iv*) the orthocentre and circumcentre if

$$P : Q : R = (b^2-c^2) \cos A : (c^2-a^2) \cos B : (a^2-b^2) \cos C$$

2. Forces of magnitudes 1, 4, 3 lb. act along the sides AB, BC, CD respectively, of a square ABCD. Show that the line of action of the resultant cuts AB produced at a distance $\frac{3}{4}$ AB from B, and find the magnitude of their resultant.

[*Ans.* $2\sqrt{5}$ lb.]

3. ABCD is a rectangle of which the adjacent sides AB and BC are equal to 3 and 4 ft. Along AB, BC, CD and DA act forces 20, 40, 50 and 80 lb. wt. ; find the resultant.

[*Ans.* 50 lb. wt. cutting AD and CD produced at E and F distant $6\frac{2}{3}$ and 5 ft. from D.]

4. Forces 1, 4, 2, 3 lb. wt. act along the sides AB, BC, CD, DA respectively of a rhombus whose acute angle (at A) is 60°. Prove that their resultant is perpendicular to AC and that it cuts AC at a point whose distance from A is 3AC.

5. ABC is an isosceles triangle whose angle A is 120° and forces of magnitude 1, 1 and $\sqrt{3}$ lb. wt. act along AB, AC and BC ; show that the resultant bisects BC and is parallel to one of the other sides of the triangle. (*P. U. 1941*)

6. Three forces P, 2P and 3P act along the sides AB, BC and CA respectively of a given equilateral triangle ABC ; find the magnitude and direction of their resultant, and find also the point K in which its line of action meets the side BC.

(*D.U. 1939*)

[*Ans.* $P\sqrt{3}$ perp. to BC. $BK = \frac{3}{2}$ BC.]

7. Forces P, P+Q, P−Q act along the sides BC, CA, AB respectively of an equilateral triangle ABC. Determine their resultant completely.

[*Ans.* $\sqrt{3}$ Q. perp. to BC meeting it at a distance P.BC/2Q from its mid. point.]

8. Forces equal to 3P, 7P and 5P act along the sides AB, BC and CA of an equilateral triangle ABC ; find the resultant.

[*Ans.* $2\sqrt{3}$ P meeting BC produced at X where $CX = \frac{3}{5}BC$ and inclined at 30° to it.]

9. Forces 2, 2, 3, 2 units act along the sides AB, CD, ED, EF respectively of a regular hexagon ABCDEF. Find the magnitude of the re ultant and prove that it acts along AB.

[*Ans.* 3 units.]

10. Forces P, 3P, 2P, 6P, 4P, 5P act respectively along the sides AB, BC, CD, DE, EF and FA of a regular hexagon of a side a. Find the magnitude and direction of the resultant and also the distance from O, the centre of the hexagon, of the point in which the line of action of the resultant meets OA.

[*Ans.* $2\sqrt{7}$P, $\tan^{-1}\sqrt{3}/5$ with OA ; $21a/2$ on AO produced.

11. The moments of a force about the vertices A, B, C of an equilateral triangle whose sides are 2 ft. long are +10, +20 and −10 in foot pound units. Find the magnitude of the force and the points where the line of action intersects AB and AC.

[*Ans.* $10\sqrt{\frac{7}{3}}$ lb. wt. ; BA produced 2 ft. from A ; mid. pt. of AC.]

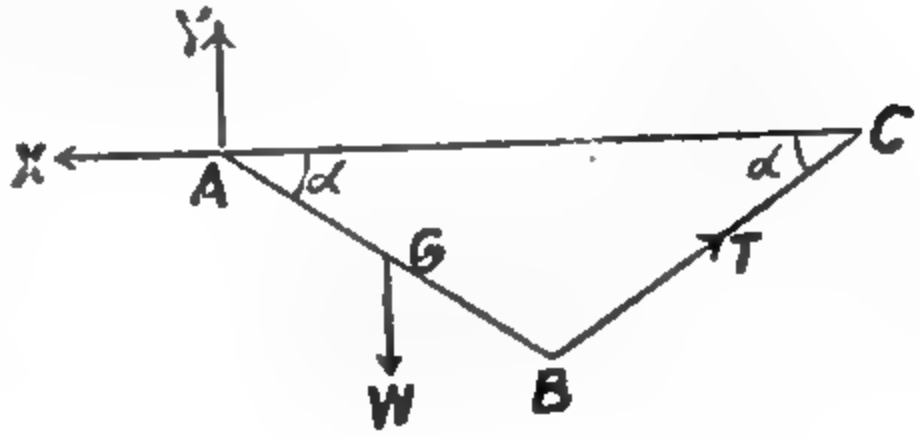
4·65. In some cases, such as those where a body is hinged to a fixed point, taking of moments about the fixed point simplifies the solution to a problem.

For, in such cases the action of the hinge, being a force passing through it, does not enter the equation of moments.

The action at the hinge is then obtained by taking its components as X, Y into two perpendicular directions and resolving forces along them.

EXAMPLES XV.

1. *One end of a uniform rod is movable about a hinge and the other is supported by a string attached to it; the string is inclined at the same angle a to the horizon as the rod. Find the reaction at the hinge.* **(P.U. 1942)**



Let $2a$ be the length and W the weight of the rod AB.

Taking moments about A, we have

$$W.AG\cos a = T.AB\sin \angle ABC$$
$$= T\,2a\sin 2a$$
$$= 4Ta\sin a\cos a$$

$$\therefore \quad T = \tfrac{1}{4}W\operatorname{cosec} a, \qquad (\because AG = a)$$

which gives the tension of the string.

Now to find the reaction, suppose its horizontal and vertical components are X and Y.

Resolving horizontally and vertically, we have

$$X = T\cos a = \tfrac{1}{4}W\cot a.$$
$$Y = W - T\sin a = W - \tfrac{1}{4}W = \tfrac{3}{4}W.$$

Hence
$$R = \sqrt{X^2+Y^2} = \tfrac{1}{4}W\sqrt{9+\cot^2 a}$$
$$= \tfrac{1}{4}W\sqrt{8+\operatorname{cosec}^2 a}.$$

If θ be the inclination of R to the horizontal,

$$\tan\theta = \frac{Y}{X} = \frac{\tfrac{3}{4}W}{\tfrac{1}{4}W\cot a} = 3\tan a.$$

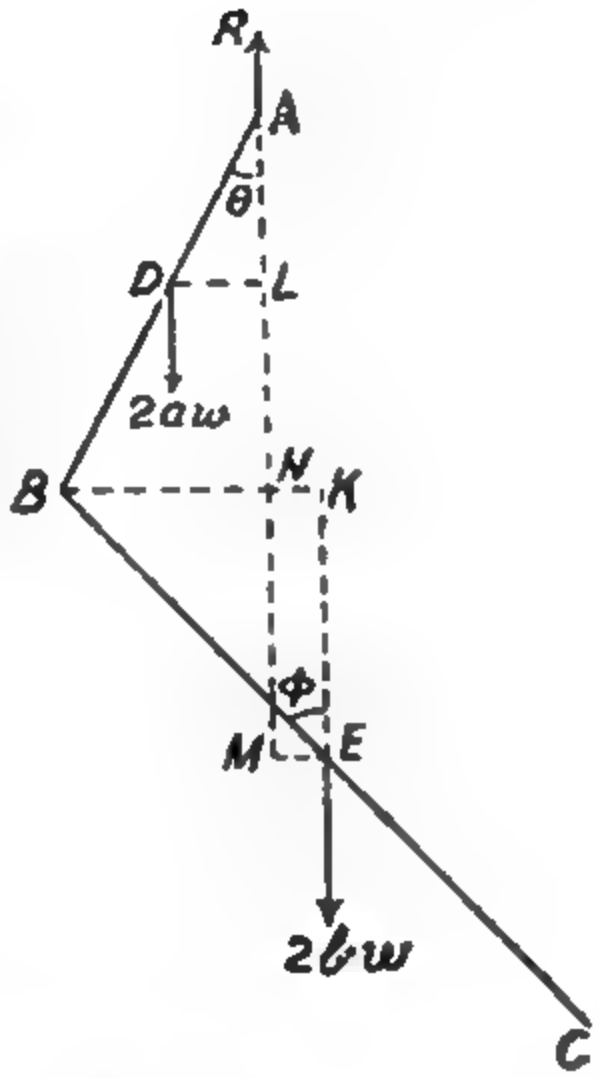
Note 1. Since the reaction R at A must pass through the point where the line of action of the tension and the weight meet, we can obtain θ also from the geometry of the figure, and then find T and R by the application of Lami's theorem.

Note 2. The method of taking moment about the hinge can be usefully applied also to the solved example given in Art. 3·4.

2. *Two uniform rods AB, BC of length 2a, 2b respectively are rigidly united at B and are suspended freely from A. If they rest inclined at angles θ, ϕ respectively to the vertical, show that*

$$\frac{\sin\theta}{\sin\phi} = \frac{b^2}{(a^2+2ab)}.$$

Let w be the weight per unit length of the rods.



Then the weight $2aw$, $2bw$ of the rods AB and BC act at their middle points D and E respectively. Since these two forces are vertical, the only other force, the reaction R at A, must also be vertical.

Taking moments about A, we have

$2aw.\mathrm{DL}=2bw.\mathrm{ME}=2bw(\mathrm{BK}-\mathrm{BN})$

$\therefore \quad a.a \sin\theta=b(b\sin\phi-2a\sin\theta)$

i.e. $(a^2+2ab)\sin\theta=b^2\sin\phi$.

Hence $\sin\theta/\sin\phi=b^2/(a^2+2ab)$.

3. Two uniform rods AB, BC, rigidly jointed at B so that ABC is a right angle, hang freely in equilibrium from a fixed point at A. The length of the rods are a and b and their weights are W_1 and W_2. Prove that,

$$\tan\theta=W_2b/(W_1+2W_2)a.$$

4. A uniform straight rod AB, of weight 12 lb. is free to turn about the end A. The rod is kept in equilibrium by a light string attached to B and to a fixed point C at the same horizontal level as A, ABC being an equilateral triangle. Find the angle which the action at A makes with the horizontal.

[*Ans*, $\tan^{-1}(3\sqrt{3})$]

5. A uniform rod AB of weight W is smoothly pivoted to a fixed point at A. The rod is maintained in the equilibrium at an angle 30° below the horizontal, by means of a force P applied at B in a direction perpendicular to AB. Find the magnitude of P, and show that the resultant reaction at the pivot is $\frac{\sqrt{7}W}{4}$.

[*Ans*. $P=\frac{\sqrt{3}W}{4}$.

6. A uniform rod AB, 12 ft. long, and weighing 50 lb. is pivoted at a point 3 ft. from A. A weight of 200 lb. is suspended from A. What force applied at B, in a direction perpendi-

cular to the rod, will keep it in equilibrium with A below B and AB inclined at 60° to the horizontal. [*Ans.* 25 lb. wt.]

7. A uniform rod ACB of weight W and length $4a$ rests upon a smooth peg C, and its upper end A is attached to a fixed point O, situated in the same horizontal line with C by means of a string OA. If AC and OA are each of length c, show that the inclination of the rod to the horizontal is $\cos^{-1}\sqrt{\frac{2a+c}{4a}}$. (*B.U.*)

8. A uniform rod AB of length 1ft. and weight 20 lb. is freely hinged to a fixed point at A. One end of an elastic string whose natural length is 18 in. is attached to B and the other end is attached to a fixed point C which is 1 ft. vertically above A. If the modulus of elasticity of the string is 80 lb. wt. prove that in the position of equilibrium, the string is streched 4·2 in. nearly.

9. A uniform equilateral triangle ABC is supported with the angle A in contact with a smooth wall by means of a string BD, equal in length to a side of the triangle, which is fastened to a point D vertically above A. Show that the distances of B and C from the wall are as 1 : 5. (*P.U. 1934*)

10. A square lamina rests with its plane perpendicular to a smooth wall one corner resting against the wall, another corner being attached to a point in the wall by a string whose length is equal to a side of the square. Show that

(*i*) a side of the square makes an angle $\tan^{-1}(\frac{1}{3})$ with the wall ;

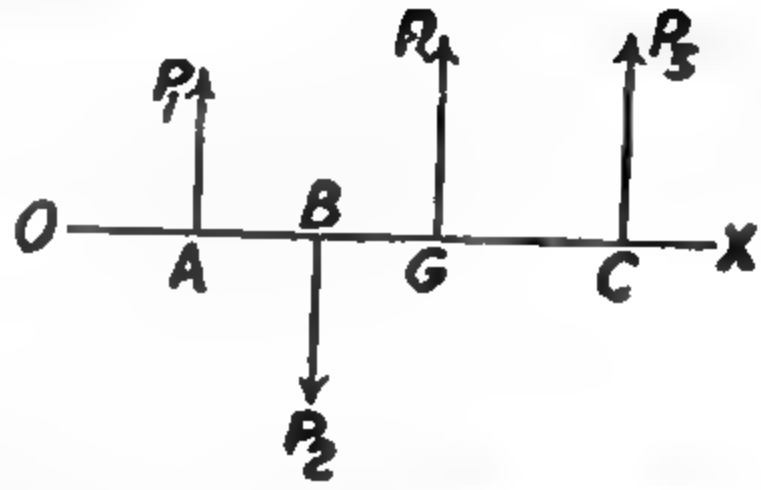
(*ii*) the tension of the string is $\frac{\sqrt{10}}{3}$ times the weight of the lamina ;

(*iii*) the distances of its angular points from the wall are as 0 : 1 : 3 : 4.

47. Position of the Centre of Parallel forces acting at fixed points on a line.

Let the parallel forces P_1, P_2, P_3.....act at the points,

A, B, C.....on the line OX, at distances x_1, x_2, x_3,.....from the point O.



Let $R = P_1 + P_2 + P_3 + \ldots\ldots$ the resultant of the forces act at G, where $OG = x$.

Since the position of the point G is independent of the direction of the forces, we suppose the forces to be perpendicular to OX.

Taking moments about O, by Art. 4·62, we get

$$(P_1 + P_2 + P_3 + \ldots\ldots)\,\bar{x} = Px_1 + P_2x_2 + P_3x_3 + \ldots\ldots$$

i.e.,
$$\bar{x} = \frac{\Sigma Px}{\Sigma P}.$$

Note. $P_1 + P_2 + P_3$.....means the algebraic sum of the forces, each force being taken with its proper sign + or −, so that ΣP may vanish in which case the forces are either in equilibrium or reduce to a couple.

4 71. Position of the Centre of Co-planar Parallel Forces acting at given points. Let OX, OY be two perpendicular axes in the plane of the forces and let the parallel forces P_1, P_2, P_3.....act at the points A_1, A_2, A_3,.....whose co-ordinates with reference to OX OY are (x_1, y_1), (x_2, y_2), (x_3, y_3),.....

Let the resultant of the forces act at G$(\bar{x}, \bar{y})$

Since the position of G is independent of the direction of the forces, we turn the forces so as to be perpendicular to OX.

Then P_1, P_2, P_3,.....intersect OX at distances x_1, x_2, x_3,.....and the resultant at a distance $\bar{x}$ from O.

Taking moments about O, we have

$$(P_1 + P_2 - P_3 + \ldots\ldots)\,\bar{x} = P_1x_1 + P_2x_2 - P_3x_3 + \ldots\ldots$$

i.e.,
$$\bar{x} = \frac{\Sigma Px}{\Sigma P}.$$

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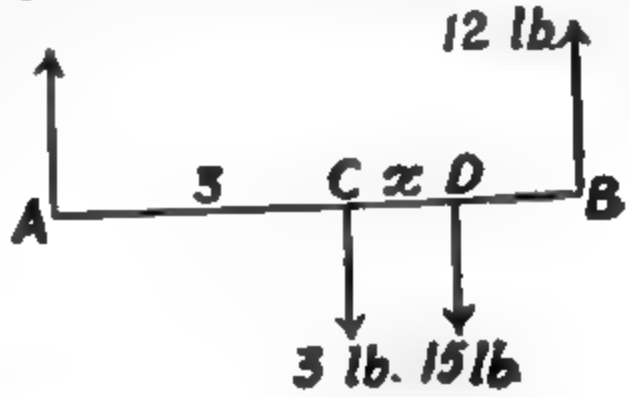
Similarly by considering forces parallel to OX.

we get $$\bar{y} = \frac{\Sigma Py}{\Sigma P}.$$

Note.—When $\Sigma P = 0$, the system is either in equilibrium or reduces to a couple.

4.72. Example (*i*). *A uniform rod of length* 6 *ft. and weight* 3 *lb. rests in a horizontal position with its ends on two supports, each of which will bear a weight of* 12 *lb. and no more. Find on what part of the rod a weight of* 15 *lb. can be placed without breaking either support.*

In the extreme case let the support at B be at the point of breaking when the 15 lb. wt. is attached at D at a distance x ft. from the centre C. The reaction at B in this case is 12 lb. wt.



Since the total weight acting downwards is $3+15=18$ lb.

$\therefore$ the reaction at $A = 18-12 = 6$ lb.

To find x, we take moments about A, we get

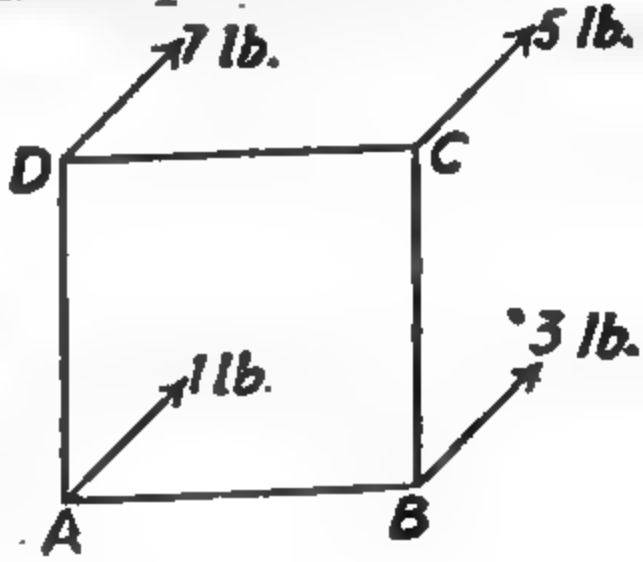
$$3\times 3 + 15\,(3+x) - 12\times 6 = 0$$

which gives $$x = 1\tfrac{1}{5} \text{ ft.}$$

Thus the weight can be placed at any point within a distance of $1\frac{1}{5}$ ft. on either side of the centre.

Example (*ii*). *Parallel forces equal to* 1, 3, 5, 7 *lb. act at the angular points A, B, C and D of a square ABCD. If the forces lie in the plane ABCD and act in the same direction find the position of the centre of these forces.*

Let x, y be the distances of the required point from the sides AD and AB and let a be the side of the square; then by Art 4.71. we have



$$x = \frac{1\times 0 + 3\times a + 5\times a + 7\times 0}{1+3+5+7} = \tfrac{1}{2}a$$

$$y = \frac{1\times 0 + 3\times 0 + 5\times a + 7\times a}{1+3+5+7} = \tfrac{3}{4}a.$$

So that the required point is at a distance $\frac{1}{4}a$ from the centre of the square.

EXAMPLES XVI

1. A uniform rod AB 5 ft. long and weighing 6 lb. has weights 1, 2, 3, 4, 5 attached to it at distances 1, 2, 3, 4, 5 ft. from the end A. Find at what point it can be supported by a single force. [*Ans.* 3·33 ft.

2. A uniform rod AB, of length 12 ft. and weighing 50 lb. rests on two supports, one at A and other 2 ft. from B. Weights of 4, 5 and 10 lb. are attached at points 2 ft., 4ft. and 8 ft. respectively from A. Find the pressures on the supports.
[*Ans* 40·8, 28 2 lb. wt.

3. A uniform shaft of weight 100 lb rests with its ends on two bearings A and B which are 10 ft. apart. Pulleys which weigh 15, 25 and 30 lb. are mounted on the shaft between A and B at distances 4, 6 and 8 ft. respectively from A. Find the pressures on the two bearings A and B. [*Ans.* 95, 75 lb. wt.

4. A heavy uniform beam AB of weight 100 lb. and length 6 ft. rests horizontaly on two fixed supports at a distance of 1 ft. from A and the other at a distance of 2 ft. from B. Find what weight should be hung from the end A to make the reaction on the two supports equal. [*Ans.* 20 lb.

5. A uniform beam AB, 18 ft. long and weight 59 lb. is freely supported in a horizontal position at points 3 ft. and 17 ft. from A. It carries loads weighing 20 lb, 17 lb. and 30 lb. at points distant 6 ft., 7 ft. and 14 ft. respectively from A. Find the pressure on each support.

A uniform bar 4 ft. long and weighing 70 lb. is placed on the beam. Prove that the pressures on the points of support will become equal if one end of the bar is over the middle point of the beam. [*Ans*, 68 lb., 58 lb.

6. A uniform horizontal bar AB, 12 ft. long, weighing 1 lb. per ft., is supported at two points C, and D. It carries load of 4 lb. and 11 lb. at E, and F respectively, where AE= 5 ft. and AF=10 ft. If AC=4 ft., find the smallest possible value of CD.

If AC. AD are respectively 4 ft. and 8 ft., what length of the bar may be cut off from the end A without disturbing the equilibrium ? [*Ans.* $3\frac{3}{7}$ ft., 2 ft.

7. ABCD is a straight rigid rod in which AB is uniform and of length a and weight w per unit length, BC is uniform and of length b and weight $2w$ per unit length, CD is uniform and of length c and weight $3w$ per unit length. If the rod rests horizontally in equilibrium on two pegs placed respectively at B and C, find the forces exerted on the pegs and show that

$$3(b+c)^2>a^2+b^2 \quad \text{and} \quad (a+b)^2+b^2>3c^2. \quad \text{(H. C.)}$$

8. Weights 1, 5, 3, 4, 2, 6, lb. wt. are placed at the angular points of a regular hexagon, taken in order ; prove that their resultant passes through the centre of the hexagon.

9. Parallel forces 1, 2, 3 lb. wt. act at the corners A, C, E of a regular hexagon AB 'DEF, of side a, and forces 4, 5, 6 lb. wt. act at the corners B, D, F in a direction parallel to the first three but in opposite sense. Find the distance of the centre of these forces from the centre of the hexagon..

[*Ans.* $a\sqrt{3}/9$.

10. Like parallel forces 6, 10, 14, 10 lb. wt. act respectively at A, B, C D, the vertices of a parallelogram. Show that the centre and resultant of these parallel forces remains the same, if instead of these forces, parall l forces 8, 12, 16 and 4 lb wt. act at the middle points of the sides AB, BC, CD and DA respectively.

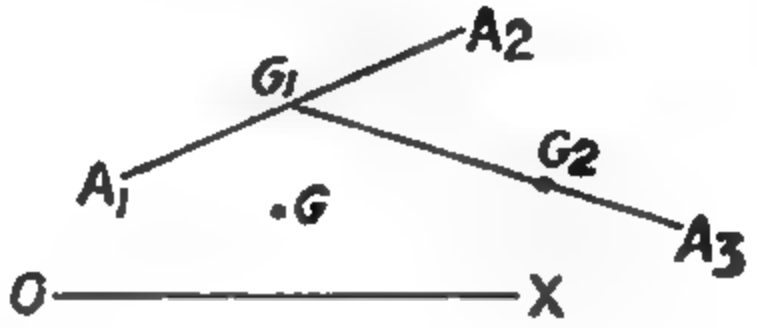
4.8 Moment of a force about a perpendicular line. The moment of a force F about a perpendicular line (in space) which does not meet it, is equal to the product of the force F, and the perpendicular distance between the given line and the line of action of the force.

4·81. Theorem VII. *If a number of parallel forces (in space) have a resultant, the sum of their moments about a line perpendicular to them is equal to the moment of their resultant.*

Let the moments be taken about OX, a line perpendicular to the given forces P_1, P_2, P_3.....Draw a plane through OX perpendicular to the forces, meeting them in the points A_1, A_2, A_3.....and their resultant in G at distances y_1, y_2, y_3.....and $\bar{y}$ respectively, from OX.

(The forces are taken perpendicular to the plane of the paper).

Let the resultant of P_1 and P_2 act at G_1 then since



$$P_1.AG_1 = P_2 G_1A_2,$$

or $$\frac{AG_1}{G_1A_2} = \frac{P_2}{P_1}$$

$\therefore$ the distance of G_1 from $OX = \dfrac{P_1y_1 + P_2y_2}{P_1 + P_2}$.

(By Co-ordinate Geometry)

Again let the resultant of $P_1 + P_2$ at G_1 and P_3 at A_3 act at G_2, then $(P_1 + P_2).G_1G_2 = P_3 G_2A_3$.

$\therefore$ the distance of G_2 from OX

$$= \frac{(P_1 + P_2)\dfrac{P_1y_1 + P_2y_2}{P_1 + P_2} + P_3y_3}{P_1 + P_2 + P_3}$$

$$= \frac{P_1y_1 + P_2y_2 + P_3y_3}{P_1 + P_2 + P_3}$$

Proceeding in this manner, we get,

$$\bar{y} = \frac{P_1y_1 + P_2y_2 + P_3y_3 + \ldots\ldots}{P_1 + P_2 + P_3 + \ldots\ldots} \qquad \ldots\ldots(1)$$

i.e, $(P_1 + P_2 + P_3 + \ldots\ldots)\bar{y} = P_1 y_1 + P_2 y_2 + P_3y_3 + \ldots\ldots$

$\therefore$ the moment of the resultant about OX $\Big\}$ = sum of the moments of the forces P_1, P_2, P_3....etc., about OX.

Cor. *If a rigid body has an axis fixed, and is acted upon by parallel forces perpendicular to the axis, it is in equilibrium if the algebraic sum of the moments of the forces about the fixed axis vanishes.*

4·82. Position of the centre of non-coplanar parallel forces acting at given points on a plane.

Let the forces P_1, P_2, P_3.....intersect at A_1 (x_1, y_1), A_2 (x_2, y_2), $A_3(x_3, y_3)$.....the plane XOY, OX, OY being the co-ordinate axes and let their resultant meet the same plane in $(\bar{x}, \bar{y})$.

The forces being parallel they make the same angle with the plane. Since the position of the centre of the parallel forces does not depend upon their direction, we may suppose them to be perpendicular to the plane XOY.

Taking moments about OX we have

$$(P_1+P_2+P_3+\ldots\ldots)y = P_1y_1+P_2y_2+P_3y_3+\ldots\ldots$$

Similarly by taking moments about OY, we get

$$(P_1+P_2+P_3+\ldots\ldots)x = P_1x_1+P_2x_2+P_3x_3+\ldots\ldots$$

Thus $\bar{x} = \dfrac{\Sigma Px}{\Sigma P}$, $\bar{y} = \dfrac{\Sigma Py}{\Sigma P}$.

4·83. Example. *Like parallel forces P, Q, R act at the vertices of a triangle ABC perpendicular to its plane. Show that if the resultant passes through the incentre, the forces are proportional to the lengths of the opposite sides.*

Let F be the resultant passing through the incentre.

Taking moments about the line BC, we have

$$F.r = P \times \text{perp. from A on BC.}$$

$$= P.b \sin C.$$

Similarly by taking moments about CA, we have

$$F.r = Q.a \sin C.$$

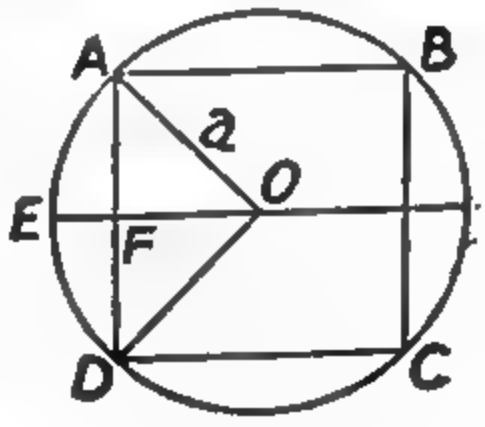
$\therefore$ $P.b = Q.a$ *i. e.* $P : Q = a : b$. Similarly $Q : R = b : c$.

Hence $P : Q : R = a : b : c$.

4·84. Toppling and Sliding.

Example. (*i*) *A circular table weighing W lb. has four legs spaced at equal distances round its edge. Find the least weight sufficient to overturn the table.*

Let the table be at the point of being overturned about the line joining the feet of legs at A and D, and let w be the required weight and a the radius.



In order that the w may be the least, it must be placed at E, the point most distant from AD.

Taking moments about the line joining the feet of the legs at A and D, we have

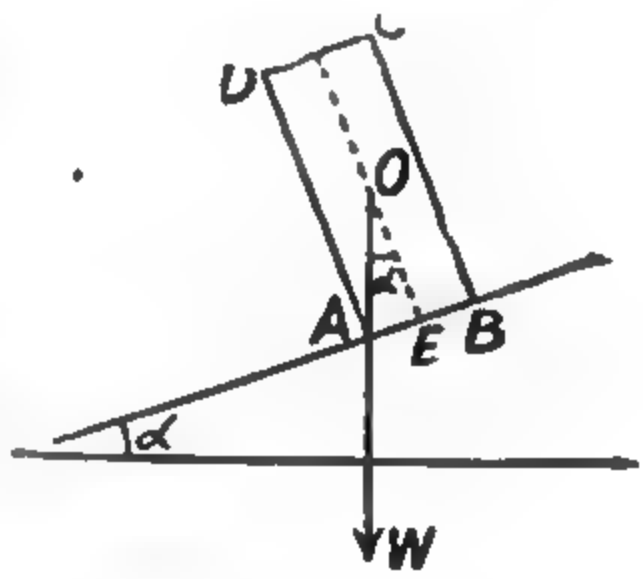
$W.FO = w.EF.$

i.e., $Wa\cos 45° = w(a - a\cos 45°)$

or $W\frac{1}{\sqrt{2}} = w\left(1 - \frac{1}{\sqrt{2}}\right)$ *i. e.* $w = \frac{W}{\sqrt{2}-1} = W(\sqrt{2}+1)$.

Example. (*ii*). *A uniform rectangular block of height h whose base is a square of side a rests on a rough horizontal plane. The plane is gradually tilted about a line parallel to two edges of the base. Show that the block will slide or topple according as a is greater or less than μh, where μ is the co-efficient of friction.*

Let ABCD be the vertical section of the block through its centre O, perpendicular to the specified edges. Let α be the inclination of the plane when the block is on the point of tumbling over.



Then since the vertical line through O, must pass through A and since $AB = a$ and $BC = h$, we have

$$\tan\alpha = \tan\angle AOE = \frac{AE}{OE} = \frac{AB}{BC} = \frac{a}{h}. \quad \ldots\ldots(1)$$

Also β, the inclination of the plane, when the block is about to slide, is given by

$$\tan\beta = \mu. \quad \ldots\ldots(2)$$

Hence the block will slide before it topples,

$$\text{if } \beta < \alpha \quad i.\,e.\ \text{if } \mu < \frac{1}{h} \text{ or if } a > \mu h.$$

EXAMPLES XVII

1. Three like parallel forces P, Q, R act at the corners of a triangle ABC, prove that for all directions of the forces their resultant passes through the

(i) circumcentre if P : Q : R = sin 2A : sin 2B : sin 2C.

(ii) incentre if P : Q : R = sin A : sin B : sin C.

(iii) orthocentre if P : Q : R = tan A : tan B : tan C.

2. A light table stands on three equal vertical legs and a weight is placed at the centre of the circle inscribed in the triangle formed by the points of intersection of the legs with the table, show that the pressures on the legs are proportional to the lengths of the opposite sides of the triangle. (*P.U. 1941*)

3. A circular table whose weight is W lb. is provided with three vertical legs attached to three points in the circumference equidistant from one another; find the least weight which when hung in the edge of the table will just cause it to overturn. (*Ans.* W lb.)

4. A circular tray of radius a stands on a single circular foot of radius b. If P is the whole weight of the tray and its support, find how far from the centre a weight Q can be placed without the tray falling over. (*Ans.* $b(1+P/Q)$ (*P.U. 1938*)

5. A force P passing through a point O within a triangle ABC acts perpendicular to the plane of the triangle If D, E, F are the points of intersection of AO, BO, CO with BC, CA AB respectively, prove that P is equivalent to P $\frac{OD}{AD}$, P $\frac{OE}{BE}$ P $\frac{OF}{CF}$ passing through A, B, C perpendicular to the same plane.

6. A triangular lamina ABC, obtuse angled at C, rests in a vertical plane with the side AC in contact with a horizontal table. Find the greatest weight that can be suspended from B without disturbing the equilibrium. The sides of the lamina are $BC=a$, $CA=b$, $AB=c$ and its weight is W.

What would happen if $c^2>a^2+3b^2$? (*P U. 1942*)

Ans. $W(3b^2+a^2-c^2)/3(c^2-a^2-b^2)$

7. In question 6, if $\angle C=120°$, show that the triangle will not topple over without any weight at B if $BC<2AC$. In case $BC<2AC$, show that the least downward vertical force applied at B which will cause the triangle to topple over is

$$\frac{W}{3}\left(\frac{2AC}{BC}-1\right).$$

8. A cubical block of wood is placed on a rough horizontal table and a horizontal force is applied to the middle of one

of the sides of the square top at right angles to that side. Prove that, as the force is increased the block will begin to slide if the co-efficient of friction is less than $\frac{1}{2}$ and that otherwise it will begin to tilt. *(P.U. 1931)*

9. A uniform solid (*i*) cylinder, (*ii*) cone of radius r and height h stands on a rough plane whose inclination to the horizon is gradually increased Show that the bodies will slide before they topple over if the co-efficient of friction is less than (*i*) $\frac{2r}{h}$. (*ii*) $\frac{4r}{h}$.

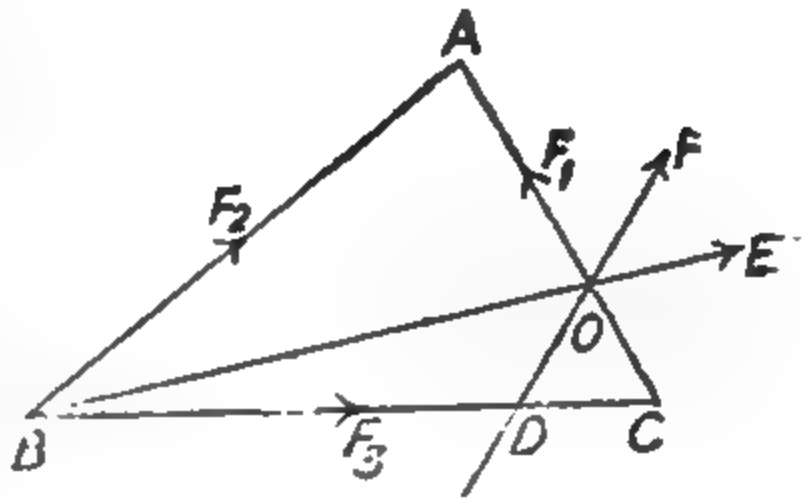
10. The cross section through the centre of a right prism is a regular hexagon ABCDEF. The prism rests with the face that contains AB on a rough horizontal table, and a gradually increasing thrust is applied at the middle point of CD perpendicular to the face that contain CD. Show that the prism will slide before it begins to turn about A if the co-efficient of friction is less than $\frac{1}{\sqrt{3}}$.

11. A triangular prism is placed with a rectangular face of breadth 2c on a rough horizontal plane, the co-efficient of friction being μ. The prism is a uniform right prism and its two inclined faces are equal rectangles whose common edge is at a hight h above the horizontal face. If the plane is rotated slowly about a line which is parallel to the three parallel edges, prove that the prism will topple or slide according as $\mu \gtrless 3c/h$.

Miscellaneous Examples II.

1. *A force F is replaced by three forces acting along the sides of a triangle ABC and its line of action meets BC internally at D, show that the ratio of the magnitude of the forces along AB and AC is CD.AB : BD.AC.*

The force F can be resolved first along CA and BO and the component along BO can further be resolved along BA and BC (Ex. 9 Mis. Ex. I P. 56).



Let F_1, F_2 be the magnitude of the forces along AC and AB acting in the directions shown in the figure. Taking moments about D, we have

$$F_1 \cdot CD \sin C = F_2 \cdot BD \sin B.$$

$$\therefore \quad \frac{F_2}{F_1} = \frac{CD \sin C}{BD \sin B}$$

$$= \frac{CD}{BD} \cdot \frac{AB}{AC} = \frac{CD.AB}{BD.AC}$$

2. Prove that if a transversal cuts the lines of action OL, OM, ON of the three forces P, Q, R in equilibrium, at the points L, M, N, then with a convention regarding signs

$$\frac{P}{OL.MN} = \frac{Q}{OM.NL} = \frac{R}{ON.LM}$$

and deduce that $\frac{P}{OL} + \frac{Q}{OM} + \frac{R}{ON} = 0$.

3. Four forces acting along the sides of a quadrilateral are in equilibrium, prove that the quadrilateral is plane. If also the quadrilateral can be inscribed in a circle, prove that each force is proportional to the opposite side. (*P. U.*)

4. A heavy wheel of weight W and radius r, is to be dragged over an obstacle of height h by a horizontal pull P applied to the centre of the wheel ; show that P must be slightly greater than $\frac{W\sqrt{2rh-h^2}}{r-h}$.

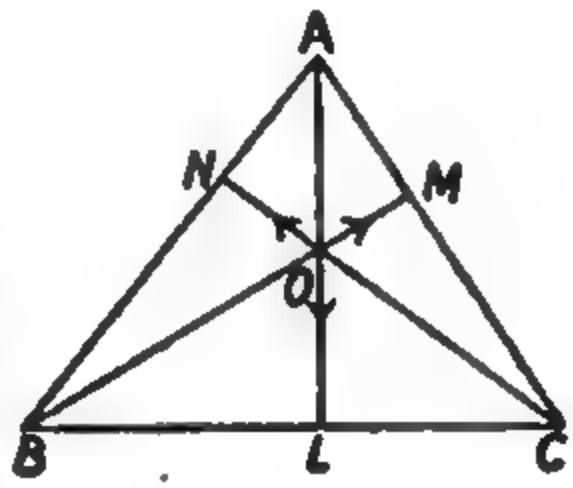
5. A hollow vertical cylinder, radius $2a$, height $3a$, rests upon a horizontal table ; a rod is placed within it with its lower end at the circumference of the base, the rod rests upon the opposite point of the upper rim and projects over. How long

must the rod be in order that it may cause the cylinder to topple over, the rod and the cylinder being of equal weight ?
[*Ans.* 15*a*.

6. *Forces P, Q, R acting along the altitudes from A, B, C of the triangle ABC are equivalent to forces X, Y, Z, acting along BC, CA, AB ; prove that*

$$\frac{X}{\cos A}=\frac{Q}{\sin B}-\frac{R}{\sin C}$$

and two similar equations.



The two systems being equivalent, the moment of the two must be the same about any point in the plane of the triangle.

Taking moments about A, through which P, Y, Z pass, we have

$$X.AL=Q.AM-R.AN$$

or $X\,b\sin C=Q\,c\cos A-R\,b\cos A$

i. e., $$\frac{X}{\cos A}=\frac{Qc}{b\sin C}-\frac{R}{\sin C}=\frac{Q}{\sin B}-\frac{R}{\sin C}$$

Two other equations are obtained by taking moments about B and C.

7. A system of forces P, Q, R, acting along the sides of the triangle ABC is equivalent to a system X, Y, Z along the sides of the pedal triangle. Prove that

$$2X=\frac{Q}{\cos B}+\frac{R}{\cos C}$$

and two other similar results.

8. A triangular lamina of weight W is supported by three vertical strings attached to its angular points so that the plane of the triangle is horizontal ; a particle of weight W is placed at the orthocentre of the triangle. Prove that the tensions of the strings are given by

$$\frac{T_1}{1+3\cot B\cot C}=\frac{T_2}{1+3\cot C\cot A}=\frac{T_3}{1+3\cot A\cot B}=\frac{W}{3}.$$

9. From four points A, B, C, D at the corners of a square in a horizontal ceiling, hang four vertical strings AP, BQ, CR, DS which support a uniform square plate PQRS of weight W, hanging in equilibrium. If λ is the modulus of each of the string AP, CR, and μ that of each of the strings BQ, DS, and if the four strings all have the same natural length l find the tension in each of them.

[*Ans.* $\frac{\lambda W}{2(\lambda+\mu)}$ in AP, CR ; $\frac{\mu W}{2(\lambda+\mu)}$ in BQ, DS.]

10. A body of weight W is supported by two vertical strings attached to points A and B in the body ; the distance AB is a, and A is lower than B. When the inclination of AB to the horizon is α then the tension in the string at A is P ; and when the inclination of AB to the horizon is β, the tension in the string at A is Q.

Prove that the perpendicular distance of the centre of gravity of the body from the line AB is

$$(P-Q)a/W(\tan\beta-\tan\alpha) \qquad (P.\ U.\ 1923)$$

11. Particles A, B of weights W_1, W_2 are suspended by light strings from a fixed point O and kept apart by a straight light rod AB. Prove that in the position of equilibrium the thrust exerted by the rod is

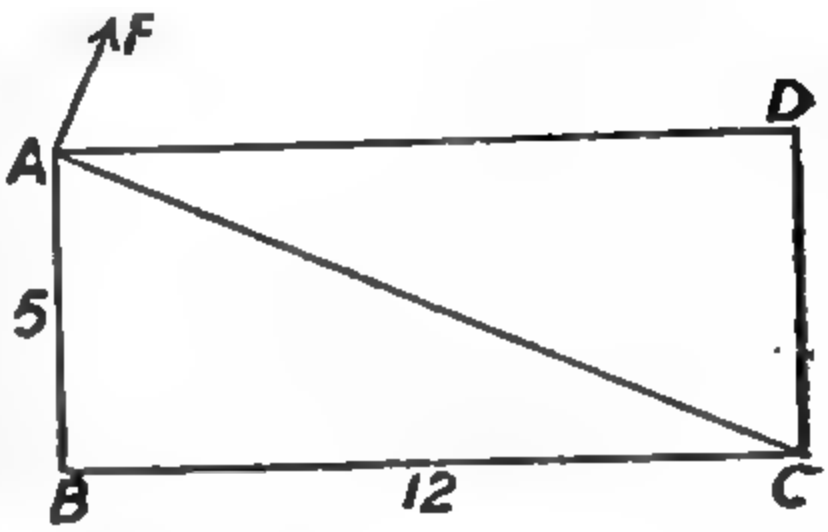
$$\frac{W_1W_2}{W_1+W_2}\cdot\frac{AB}{OG}. \qquad (L.\ U.)$$

12. ABCDE is a regular pentagon. Forces 3, 24, 24 and 16 lb. wt. act along the sides AB, AC, AD, EA respectively ; show that their resultant is 41 lb. wt. (*B. U.*)

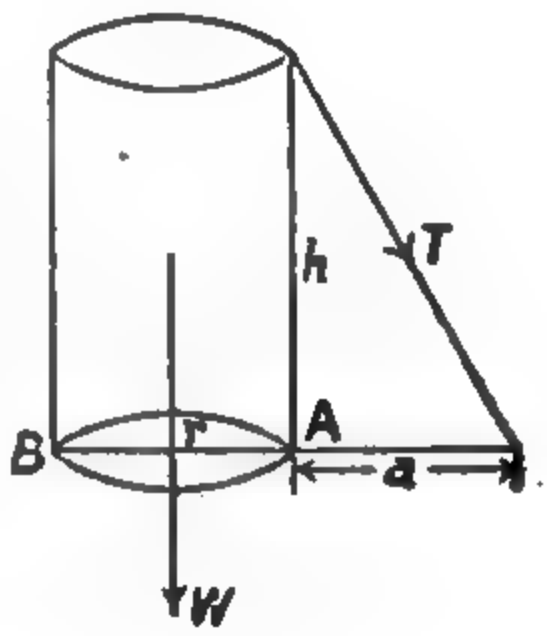
13. A heavy cube of weight 100 lb. rests on a rough floor on which it cannot slip. Prove that the least force required just to raise one edge of it off the floor is about $35\frac{1}{3}$ lb., and find where it must be applied.

[*Ans.* At the mid-point of the edge vertically above the one raised).

14. ABCD is the vertical section (through the centre) of a rectangular block resting on a rough horizontal plane. The end AB is slightly raised by a force F applied at A at right angles to AC. If AB =5 ft and BC=12 ft., show that the edge C will slip if the co-efficient of friction is less than $\frac{30}{97}$.



15. A cylinder of radius r and height h, is placed with its axis vertical on a rough horizontal table, the shortest distance between the base of the cylinder and the edge of the table being a. A string is attached to the top of the cylinder and passes over the edge of the table with a weight hanging freely tied to its lower end. If this weight is gradually increased, prove that the cylinder starts to slip before it topples over, if the co-efficient of friction is less than



$$\frac{ra}{h(r+a)}.$$

16. A homogeneous solid cube stands on a rough horizontal plane, the co-efficient of limiting statical friction between the cube and the plane being μ. A horizontal force P is applied at the centre of, and prependicular to, one of the vertical faces of the cube.

(*i*) If P is so small that the equilibrium is not disturbed, prove that the resultant normal reaction of the plane on the cube cannot pass through the centre of the cube.

(*ii*) If $\mu>1$, prove that as P is gradually increased the equilibrium will be broken before the friction becomes limiting

17. A uniform bar AB of weight W lies horizontally upon two equally rough fixed horizontal bars at right angles to it at A and at a point C between A and B, where AB, $=2a$, AC$=b$ and $b>a$. Find the reactions at A and C.

If a horizontal force P is applied to the bar at B at right

angles to BA, find expressions for P when the bar is just about to slip (*i*) at A, (*ii*) at C.

Hence show that if P were gradually increased from zero, the bar will slip at A or at C first, according as b is less than or greater than $\frac{4}{3}a$. (*H. C.*)

[*Ans.* W $\frac{b-a}{b}$, W $\frac{a}{b}$; μ W $\frac{b-a}{2a-o}$; $\frac{1}{2}\mu$W.]

18. A uniform rod of length l and weight W lies on a rough horizontal plane, the co-efficient of friction being μ. A string is attached to one end A and is pulled horizontally in a direction perpendicular to the rod so that the tension gradually increases. Show that the rod begins to turn about a point $l/\sqrt{2}$ from A and that the tension of the string is $(\sqrt{2}-1)\,\mu Wl$.

REVISION QUESTIONS. II

1. Two unlike parallel forces P and Q act at the points A and B. If their resultant R acts at C, a point in AB, prove that

(*i*) R is parallel to P or Q,

and (*ii*) $\frac{P}{BC}=\frac{Q}{AC}=\frac{R}{AB}$. (*D. U.* 1942)

2. Parallel forces P_1, P_2.....P_n act in a plane at points A_1, A_2,.....A_n whose co-ordinates referred to rectangular axes are (x_1, y_1) ; (x_2, y_2)..... (x_n, y_n). Find the co-ordinates of their centre.

Three parallel forces P, Q, R act at the vertices A, B, C, of a triangle and are proportional respectively to a, b, c, the sides of the triangle. Show that their centre coincides with the incentre of the triangle. (*P. U.* 1940)

3. If a rigid body is in equilibrium under the action of three forces two of which are parallel, the third force must be also parallel to the other two and must lie in their plane.

A circular metallic plate of uniform thickness and of weight w, is hung from a point in its circumference. A string wound on its edge carries a weight p. Find the angle which the diameter through the point of suspension makes with the vertical. (*P. U.* 1934)

[*Ans.* $\sin^{-1}\frac{p}{p+w}$

4. A body of weight W is hung by two vertical strings from two points A and B in it. When the tension in the string at A is iucreased by an amount T, the inclination of AB to the vertical changes from θ to ϕ. Show that the distance of the centre of gravity of the body from AB is

$$\frac{T}{W} \cdot \frac{AB}{\cot\theta - \cot\phi}$$ (*All. U.*)

5. A uniform circular disc of weight nW has a particle of weight W, attached to a point on its rim. If the disc be suspended from a point A on its rim, B is the lowest point ; also if suspended from B, A is the lowest point. Show that the angle subtended by AB at the centre of thedisc is

$$2\sec^{-1} 2(n+1).$$ (*P. U.* 1932)

6. Prove that the sum of the moments of a number of parallel forces about a line perpendicular to them is equal to the moment of their resultant.

A heavy horizontal circular ring of weight W rests on three supports at the points A, B and C of its circumference. Find the reactions of the supports in terms of the angles of the triangle.

[*Ans.* pressure at A$=$W cos A/2 sin B sin C.]

7. Two children each of weight W, are swinging on a see-saw formed by placing a plank of weight w across a horizontal cylinder of radius a. When the plank has been turned through an angle θ with the horizon, show that the moment of the forces tending to turn it back is

$$(2W+w)\, a\, \theta \cos\theta.$$

8. A rough circular cylinder of radius a, is fixed with its axis horizontal and a uniform rod of mass M and length $2b$ rests on it in a horizontal position perpendicular to the axis. If now a particle of mass m be attached to the rod at one end show that the rod may be allowed to turn slowly, without slipping, into a position of rest inclined to the horizontal but still perpendicular to the axis of the cylinder, provided that the angle of friction between the rod and the cylinder exceeds $mb/(M+m)a$.

9. Given the moments M_1, M_2 of a force R about two given points A and B, show that the line of action of the force passes through a fixed point C on the line AB.

Find, geometrically, the line of action of the force.

10. Find the locus of all the points in a plane such that two concurrent forces P, Q shall have equal moments (in the same sense) about any one of these points.

CHAPTER V.

COUPLES, FORCES ACTING ON A RIGID BODY

Couples

5·1. A system of two equal, unlike parallel forces is called **a couple.**

The distance between the parallel forces forming a couple is called the **arm** of the couple.

A couple each of whose force is P and the length of whose arm is p, will be denoted as (P, p).

5.2. Moment of a couple. *The algebraic sum of the moments of the forces forming a couple about any point in its plane is constant.*

Through any point O in the plane of the forces draw a straight line perpendicular to the line of action of the forces meetting them in A and B, then

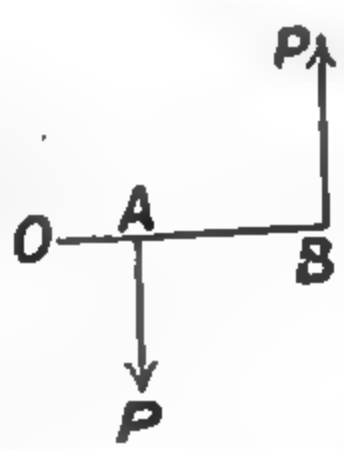


Fig. 1

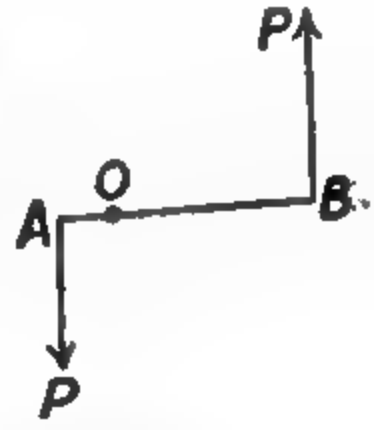


Fig. 2

Sum of the moments of the forces about O—

Fig. I

$=P.OB-P.OA$

$=P(OB-OA)$

$=P.AB$

Fig. II

$=P.OB+P.AO$

$=P(AO+OB)$

$=P.AB.$

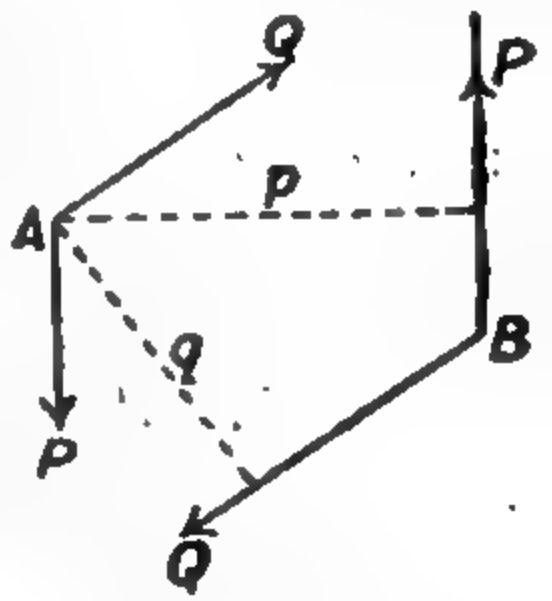
which is constant, being independent of the position of O, and is known as the *moment of the couple.*

Cor. The moment of a couple is equal to the moment of one of its forces about any point on the line of action of the other.

53. Equilibrium of two couples. *Two couples in the same plane of equal and opposite moments, balance.*

Case 1. *When the forces forming the couples are not all parallel.*

Let one of the forces P of one of the couples intersect one of the forces Q of the other in A and let the other two forces meet in B. also let p, q be the arms of the couples.



Since the moments of the couples are equal and opposite.

$$\therefore \quad P.p - Q.q = 0 \qquad \ldots(1)$$

Now, the sum of the moments about A of P and Q acting at B $= P.p - Q\,q.$

$$= 0 \qquad \text{[from (1)}$$

Hence the resultant of P and Q at B passes through A, and therefore acts along BA.

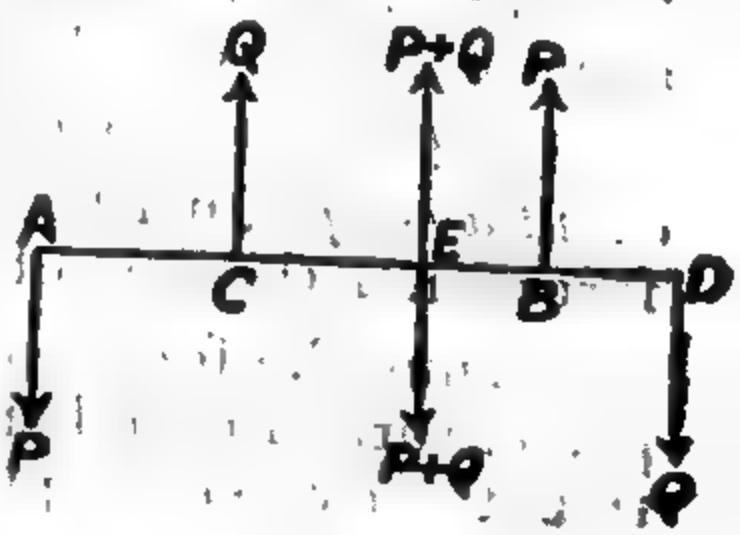
Similarly the resultant of P and Q at A acts along AB.

But since forces P and Q at B are equal and opposite respectively to P and Q at A and these pairs act at the same angle, their resultants must be equal in magnitude and opposite in sense ; and acting as they do along AB and BA, they cancel each other.

Hence the resultant of all the four forces forming the couples is nil and the couples balance.

Case II. *When the forces forming the couples are all parallel.*

Draw a straight line perpendicular to the lines of action of the forces meeting them at A, B, C and D.



Given $P.AB - Q.CD = 0 \quad \ldots(1)$

Let the resultant of P at B and Q at C act at E.

then $P.EB = Q.CE$ ...(2)

From (1) and (2), we have

$$P(AB - EB) = Q(CD - CE)$$

i.e., $P.AE = Q.DE$.

So that the resultant of P at A and Q at D also acts at E.

∴ the resultant of P at B and Q at C is P+Q acting at E in a direction parallel to that of P at B (or Q at C) and also the resultant of P at A and Q at D is P+Q acting again, at E in a direction parallel to that of P at A (or Q at D).

These two forces being equal, opposite and collinear are in equilibrium and hence the two couples balance.

5 31. *A simpler proof of the proposition of Art. 5·3 is as follows* :

The moments of the couples being equal and opposite, it follows that the sum of the moments of the four forces of the couples about *any* point in their plane must vanish, so that their resultant cannot be a couple.

The resultant cannot also be a force, for if it were so, the sum of the moments of the forces about every point being zero, the line of action of the resultant force has to pass through every point in the plane which is obviously impossible.

Hence the two couples balance.

5 32. Equivalence of two couples. *Any two coplanar couples of equal moments in the same sense, are equivalent to one another.*

Since two couples in the same plane of equal and opposite moments balance, we have by reversing the directions of the forces composing one of the couples, that two couples in the same plane of equal moments are equal.

It follows that *the moment is the only quantity of importance about a couple*, neither position, magnitude or direction of the forces, nor the arm of the couple being separately of importance.

Thus the forces of a couple can be moved to any part of its plane. turned round through any angle, and altered in magnitude to any extent, provided only that they are kept equal and

parallel, so as still to form a couple, and that the arm is correspondingly altered to keep the moment unchanged.

Example. *Forces forming a couple are 8 lb. each and the arm of the couple is 3 ft. Find the force of an equivalent couple whose arm is 5 ft.*

Let X be the required force then since the moments of the couples are equal, we have

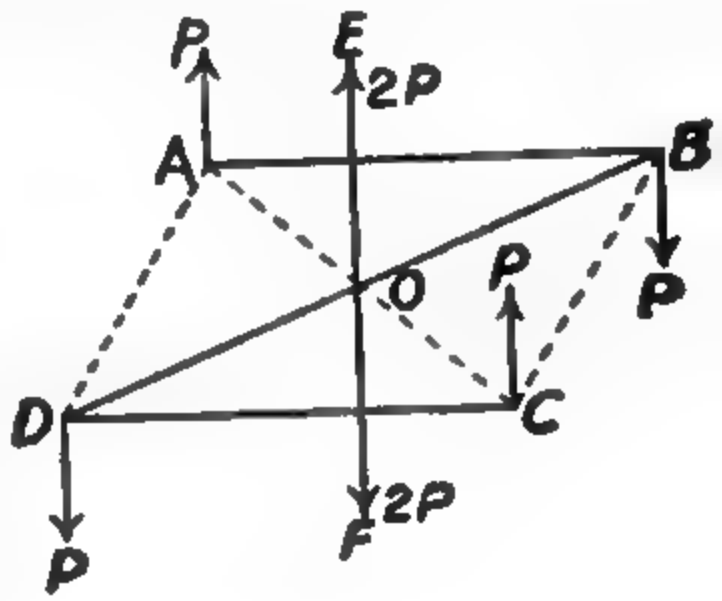
$$X \times 5 = 8 \times 3 \quad i.e., \quad X = \frac{8 \times 3}{5} = 4\frac{4}{5} \text{ lb.}$$

5.4. Couples in Parallel Planes. *Two couples in parallel planes, of equal and opposite moments, balance.*

Let one of the couples consist of forces P and arm AB, situated in a plane, say p_1.

Let the other couple of equal and opposite moment be situated in a parallel plane, say p_2.

Alter the forces of this second couple so as to make them equal and parallel to those of the first, its arm CD being thus equal and parallel to AB.



Then ABCD is a parallelogram (in a plane different from the planes p_1 and p_2) and the diagonals AC and BD bisect each other in O.

Draw EOF in a direction parallel to the direction of the forces.

Now the force P at A and P at C are equivalent to 2P at O along OE, and the forces P at B and D are equivalent to 2P at O in the opposite direction OF. Being equal and opposite and collinear these forces balance and hence the couples balance, so that two couples of equal and opposite moments in parallel planes are in equilibrium.

By reversing the directions of the forces of one of the two couples, it is easy to see that *two couples of equal moments in parallel planes are equivalent.*

Note. Obviously by Art. 5.4, this theorem holds true even when the couples are in parallel planes.

5.51. Since the moment of a couple is constant and is equal to the sum of the moments of its forces about a point, it follows that the moment of the resultant couple is equal to the algebraic sum of the moments of all the forces, forming the various couples, about any point in their plane.

EXAMPLES XVIII

1. ABCD is a rectangle with AB=4ft., BC=3 ft. ; along AB, BC, CD, DA and AC act forces 2, 7, 6, 10, and 5 lb. wt. respectively. Show that they are equivalent to a couple whose moment is equal to 46 ft. lb.

2. ABCD is a rectangle such that AB=CD=a and BC=DA=b. Forces P act along AD and CB and forces Q act along AB and CD. Prove that the perpendicular distance between the resultant of the forces P, Q at A and the resultant of the forces P, Q at C is

$$\frac{Pa-Qb}{\sqrt{P^2+Q^2}}.$$ (*P. U.* 1929)

3. ABCD is a framework of four rods in the form of a rhombus of sides l and angle α. A square CEFD of rods is constructed on CD so as to lie on that side of CD which is opposite to AB. In each of the rods AB, FE, DA, BC, DF, EC acts a force P and in the rod CD acts a force 2P, the direction of forces being indicated by the order of the letters. Show that the system reduces to a single couple of moment

$$2lP(1-\sin\alpha).$$

4. A square metal plate ABCD of side 2 ft., lies on a smooth horizontal plane and is free to turn about a point at the corner A. The plate is acted upon by forces of 10 lb. wt. along AB, 15 lb. wt. along BC and $10\sqrt{2}$ lb. wt. along DB and by a couple so adjusted that the plate is in equilibrium. Calculate the moment of the couple.

[*Ans.* 10 lb. ft.]

5. A straight uniform rod AB of length 12 ft. and of weight 10 lb. has its centre of gravity at C and is inclined at

45° to the horizontal. It is acted upon by an upward vertical force of 20 lb. wt. at the lower end A. by a downward vertical force of 10 lb. wt. at the higher end B, and by such horizontal forces at A and C that the rod is in equilibrium. Prove that the forces at A and C form a couple and find its moment.

[*Ans.* 90 $\sqrt{2}$ lb. ft.]

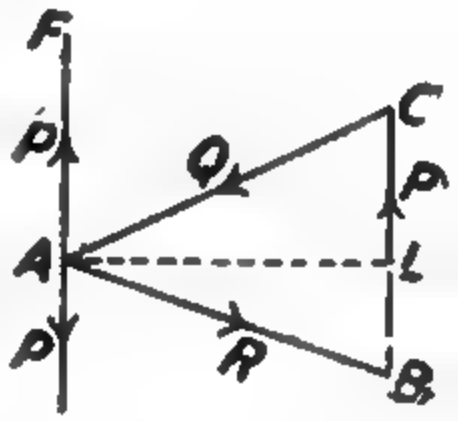
6. ABC is a triangle whose sides BC, CA, AB are 25, 15 and 20 inches respectively. A force of 25 lb. wt. acts along the bisector of the angle BAC; parallel forces through B and C are in equilibrium with this. If the forces are now turned about A, B and C, till they are at right angles to BC, prove that they form a couple of moment $3\frac{4}{7}$ lb. ft.

7. The algebraic sum of the moments of two given coplanar forces about *any* point in their plane is constant. Show that the given forces form a couple.

5.6. Theorem IX. *Three forces represented in magnitude, direction and line of action by the sides of a triangle taken in order, are equivalent to a couple.*

Let the forces P, Q, R acting along the sides BC, CA and AB of the triangle ABC, be represented by these sides.

Draw EAF parallel to BC. At A introduce two equal and opposite forces, each equal to P.



The forces P, Q, R *acting at A* along AF, CA and AB are in equilibrium by the triangle of forces.

We are thus left with two unlike parallel forces each equal to P, one acting along BC and the other acting along AE. They evidently form a couple.

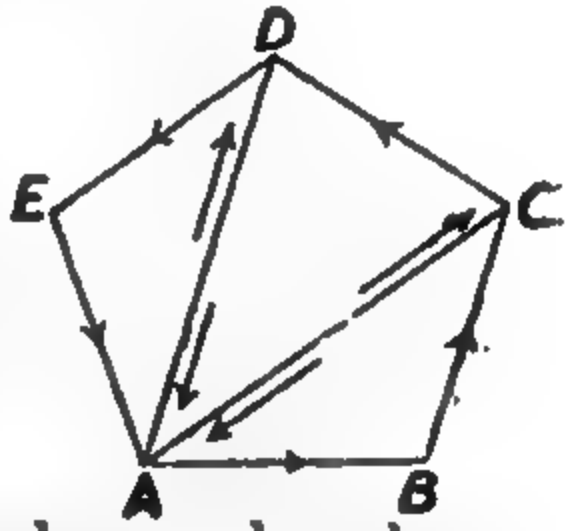
The moment of the couple = P.AL = BC.AL
= twice the area of the $\triangle$ ABC.

Ex. Show that the resultant of a system of three forces acting along the sides of a triangle is either a couple or a single force.

5.61. *Forces represented in magnitude, direction and line of action by the sides of a plane polygon are equivalant to a couple whose moment is represented by twice the area of the polygon.*

This follows at once from the above theorem.

Divide the polygon into triangles by joining one corner to each of the other, to which it is not already joined. Introduce equal and opposite forces represented completely by the lines so drawn. Thus we obtain a number of sets of three forces acting along the sides of a triangle and represented by them. Applying the above theorem to each of these sets, we obtain a number of co-planar couples which by Art. 5·5 are together equivalent to a single couple whose moment is equivalent to twice the sum of the areas of the triangles *i.e.*, twice the area of the polygon.



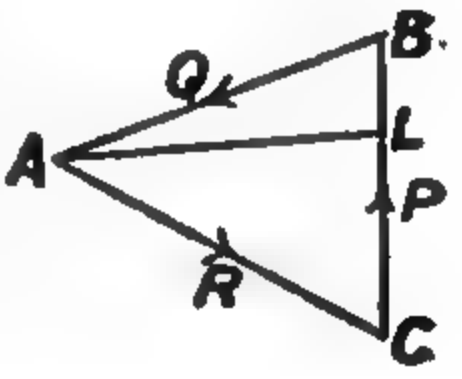
Note. The effect of such a system of forces acting upon a *rigid body* is a simple *rotation* and no translation (the sum of the resolved parts of the forces *i.e.*, of the sides of the polygon. vanish in all directions).

If, how ver, tho forces act on a *particle* and not on a rigid body, the effect of the forces is nil, (by the theorem of polygon of forces).

5·62. *If three co-planar forces, no two of which are parallel, are equivalent to a couple, they can be represented by the sides of the triangle formed by their lines of action.*

Let P, Q, R be the forces and ABC the triangle formed by their lines of action.

Let G be the moment of the couple and Δ the area of the Δ ABC, then since the moment of a couple can be measured in terms of the area of a triangle, we can suppose $G=2\Delta$, on some scale.



Taking moments about A, we have

$P.AL=G=2\Delta=BC.AL$ *i.e.*, $P=BC$

Similarly $Q=CA$ and $R=AB$.

Thus the forces are represented by the sides of the Δ ABC.

EXAMPLES XIX

1. Four forces are completely represented by the sides BA, AD, CB, DC of a quadrilateral ABCD. Show that they are

equivalent to a couple consisting of two forces through A and C, each equal and parallel to the other diagonal.

Show also that the moment of the couple is represented by four times the area of the figure formed by joining the middle points of the sides.

2. ABC is a triangle and forces represented by l BC, m.CA, n.AB act along the sides BC, CA, AB respectively. Show that if $l=m=n$, the system is equivalent to a couple, and if $l+m+n=0$ the resultant passes through G, the centroid of the triangle.

3. P, Q, R are taken on the sides BC, CA, AB respectively of the triangle ABC, dividing each in the ratio of $1+\lambda : 1-\lambda$ in the same sense round the triangle. Prove that the forces represented by AP, BQ, CR are equivalent to a couple. Find the moment of the couple. [*Ans.* $2\lambda\triangle$.]

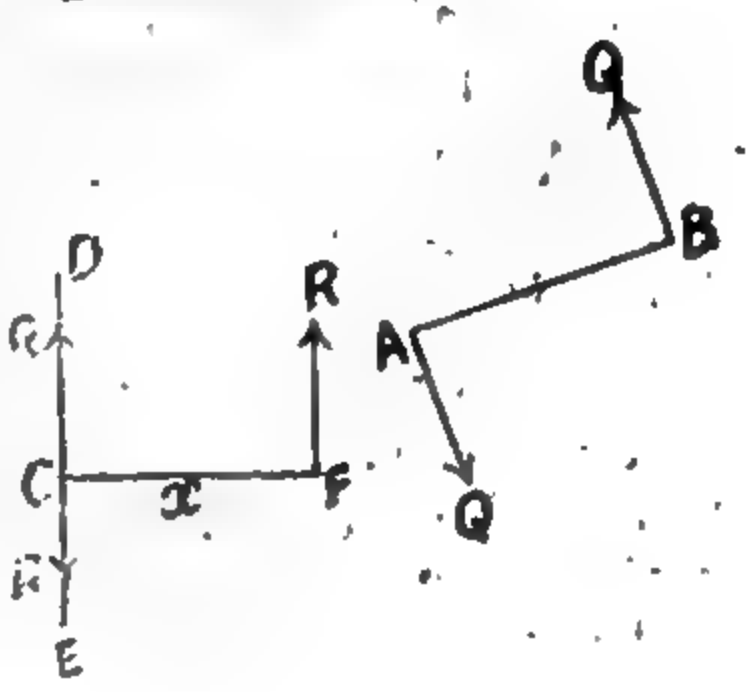
4. X, Y, Z are points in the sides BC, CA, AB of the $\triangle$ ABC such that $\frac{BX}{XC} = \frac{CY}{YA} = \frac{AZ}{ZB} = \frac{\mu}{\lambda}$. Prove that the system of forces represented by AX, BY, CZ is equivalent to a couple of moment $\frac{\lambda-\mu}{\lambda+\mu} \cdot 2\triangle$, where $\triangle$ is the area of the triangle. (Compare Q. 10 Ex. VII.)

5. OABC is a tetrahedron. Show that the couples represented by the areas OBC, OCA, OAB can have a couple represented by the area ABC as resultant. (*P. U.* 1938)

6. ABC is a triangle and A′ B′ C′ is another whose area is $\triangle$ and whose sides B′C′, C′A′, A′B′ pass through A, B, C respectively. Forces Ka, Kb, Kc, act at A′, B′, C′ parallel to BC, CA, AB respectively. Show that they reduce to a couple whose moment is equal to $2K\triangle$.

57. Resultant of a Couple and a Co-planar Force.

Let a couple be formed by two forces, each equal to Q acting at A and B, AB=q being the arm of the couple; and let R be a force, acting in the plane of the couple at C along CD.



The moment of the couple=Q.q =G (say).

Now replace the couple by a new couple having its forces, each equal to R. If x be the arm of this new couple, then $G=Q.q=R.x$ or $x=\frac{Q.q}{R}=\frac{G}{R}$.

Shift the new couple such that one of the forces R acts at C along CE (DC produced), a direction opposite to that of the given force R at C.

Now the two forces acting at C balance and we are left with a single force R at F, where $CF=x=\frac{G}{R}$.

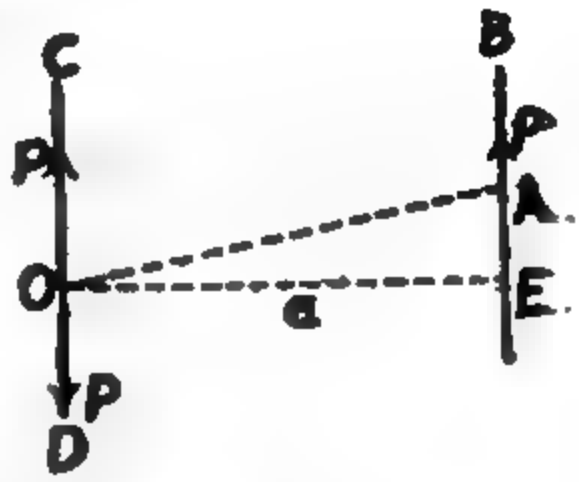
Hence a couple of moment G and a given force R compound into a single parallel force R displaced to a distance $\frac{G}{R}$ *from its original position.*

It may be noted that the magnitude and direction of the force remains independent of the couple.

Cor. *A couple and a force in the same plane cannot be in equilibrium.*

57·1. Theorem X. Resolution of a force into a force and a couple. *A force can be replaced by an equal and parallel force acting at an arbitrarily chosen point together with a couple,* (both lying in the plane determined by the given force and the chosen point).

Let P be the given force acting at a point A along a certain line AB and let O be the chosen point.



At O introduce two equal and opposite forces, each equal and parallel to the given force P.

Now the original force P at A and the force P at O, acting along OD from a couple whose moment is Pa where a is the length of the perpendicular from O on AB. Besides this, there is force P acting along OC equal and parallel to the original force P.

Thus, a force P acting at A is equivalent to a force at O equal and parallel to the original force and a couple whose moment is equal to the moment of the original force about O.

5·72. Resultant of a system of coplanar forces acting on a rigid body. The above theorem can easily be extended to any number of coplanar forces, so that they can be replaced by a number of coplanar couples and a number of concurent forces at O.

By Art. 5·5 the system of couples is equivalent to a single couple of moment say G and by Art 3·23 the forces at O are equivalent to single force, say R.

Thus we find that a *system of coplanar forces is equivalent to a single force acting at any chosen point in the plane, together with a couple in the same plane.*

The resultant force R and the resultant couple G (if none of them is zero) can further be compounded by Art. 5·7 into a single force R with its line of action not passing through O.

If, however, $G=0$ and R does not vanish, the system reduces to a single force passing through O.

Further, if $R=0$ and G does not vanish, the system is equivalent to a couple.

Thus the whole system is reducible to either a single force or a couple, a fact which was otherwise shown in Art. 4·61.

5·73. Again if G=0 as well as R=0, the couple and the single resultant both disappear, and the given system is in equilibrium. These conditions are evidently satisfied when

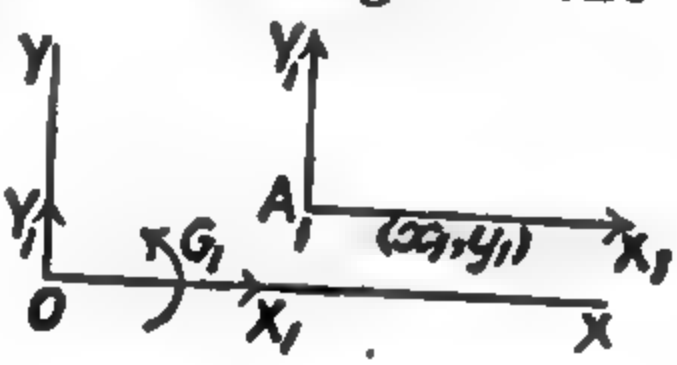
(*i*) the sum of the moments of the forces about a point in their plane vanishes ; and

(*ii*) the sum of the resolved parts of the forces along two lines separately vanish.

The above results can also be obtained analytically as follows :—

5·74. Analytical method for the resultant of a system of coplanar forces.

Let P_1, P_2,.....etc., be the given forces acting at the points, A_1, A_2,.....etc., the co-ordinates of A_1, A_2,.....being (x_1, y_1), (x_2, y_2),..... with reference to rectangular axes OX and OY in the plane of the forces.



Let X_1, Y_1 be the resolved parts of P_1 parallel to OX, OY.

Now Y_1 at A_1 can be replaced by Y_1 at O along OY, and a couple of moment x_1Y_1.

X_1 at A_1 can be replaced by X_1 at O along OX and a couple of moment $-y_1X_1$.

Hence the force P_1 at A is equivalent to

(1) X_1 along OX

(2) Y_1 along OY,

and (3) a couple of moment $x_1Y_1-y_1X_1$, say G_1.

Replacing other forces in the same manner and adding up for all the forces, we have

(1) $X=\Sigma X_1$ along OX

(2) $Y=\Sigma Y_1$ along OY,

and (3) $G=\Sigma(x_1Y_1-y_1X_1)$, a couple.

Hence ΣX_1 and ΣY_1 are the sums of the resolved parts of the given forces along OX and OY respectively and G is the sum of the moments of the forces about O.

ΣX_1, ΣY_1 are further equivalent to a single force R acting at O.

The resnltant is simply a couple if $\Sigma X_1 = 0$ and $\Sigma Y_1 = 0$ and $G \neq 0$. The moment of the couple is obtained by taking the sum of the moments of the forces about any point in their plane.

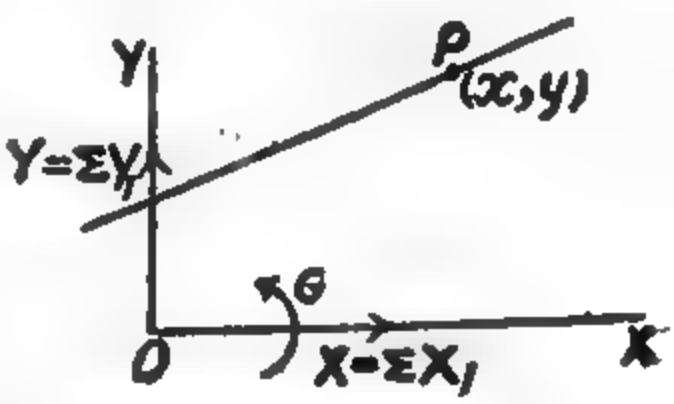
The resultant is a single force if at least one of ΣX_1 or ΣY_1 is not zero. (This includes the case when none of the three is zero).

Thus the system is reducible to either a single force or a couple.

If ΣX_1, ΣY_1 and G all the three vanish, the system is in equilibrium.

5 75. Equation of the line of action of the resultant.

In case the forces reduce to a single force, the equation of its line of action is obtained as follows :—



Let (x, y) be a point on the line of action of the resultant force, then since the resultant passes through this point the sum of the moments of the forces about it must vanish.

Hence taking moments about P, we have

$$yX - xY + G = 0 \text{ or } xY - yX = G \quad \ldots\ldots(1)$$

which is the equation of the required line.

It may be noted that the sum of the moments of all the forces of the system about any point (x, y) in their plane is

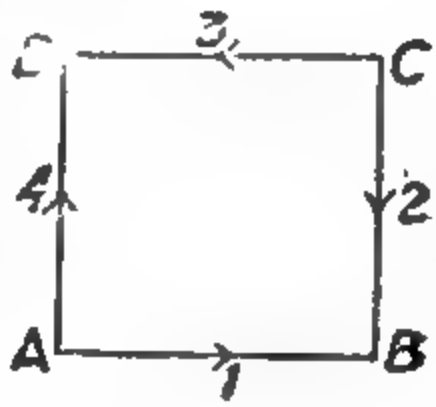
$$G - xY + yX \quad \ldots\ldots(2)$$

and the angle θ which the resultant makes with OX

$=$slope of the line $(1) = Y/X$.

5.76. Example. *Find the resultant of forces* 1, 2, 3, 4. *lb. wt. acting as shown along the sides of a square of side* 2 *ft.*

Taking AB, AD as axes of reference, we have



X=sum of the components along AB
=1−3=−2 lb.

Y=sum of the components along AD
=4−2=2 lb.

G=sum of the moments about A
=3×AD−2×AB=(3−2)2=2 lb. ft.

Hence the equation of the line of action of the resultant is

$$xY-yX=G \text{ i.e., } 2x+2y=2$$

or

$$x+y=1$$

which obviously intersects the lines AB and AD at their middle points.

Also the magnitude of the resultant $=\sqrt{X^2+Y^2}=2\sqrt{2}$ lb. and θ, the angle which the resultant makes with AB is given by

$$\tan\theta=\frac{Y}{X}=\frac{2}{-2}=-1 \text{ i.e., } \theta=135°.$$

EXAMPLE XX

1. Four coplanar forces act on a rigid body, their lines of action being the sides of a square ABCD, their magnitudes and directions are (1) 20 lb. wt. along AB (2)—10 lb. along BC, (3) 30 lb. along CD, (4)—40 lb. along DA. Find the forces, P and Q whose lines of action are respectively CA and a line through the point B which will just produce equilibrium.

(*P.U.* 1931)

[*Ans.* P=10√2 lb., Q=20√2 lb, acting along DB.

2. ABCD is a rectangle whose side AB=15″ and AD=8″. Taking AB, AD as co-ordinate axes, show that the equation of the line of action of the resultant of the following forces is $x+y=1$.

22 lb. from D to A ; 10 lb. from A to B ; 51 lb. from D to B ; 34 lb. from A to C ; 40 lb. from C to B ; 15 lb. from C to D.

3. Forces of 3, 4 and 7 lb. act along the sides AB, CB and CA of an equilateral triangle, Taking A as origin and CA as X-axis, show that the resultant acts along the line

$$7x-5\sqrt{3}y+4a=0$$

a being the length of a side of the triangle.

4. Components, parallel to the axes of co-ordinates, of four forces acting at the points (1, 2), (−2, 3), (−3, −3), (3, −4) are respectively (−3, 4). (5, 3), (−2, −3) and (4 −1), the distances being in feet and the forces in pounds weight. Find the magnitude and the direction of the single force and the moment of the couple at the origin. Prove that the line of action of the single resultant is $3x-4y=5$. (*P.U.* 1928)

5. ABCDEF is a regular hexagon and O is its centre. Forces of magnitude 1, 2, 3, 4, 5, 6 act along the lines AB, CB, CD, ED, EF, AF. Reduce the system to a force at O and a couple, and find the point in AB through which the single resultant passes.

[*Ans.* Resultant $=2\sqrt{3}$ bisects AB and AF.

6. The moments of a given system of forces about three points (2,0), (0,2) and (2,2) are +3, +4, +10 units respectively, find the magnitude of the resultant force and prove that it acts along the line $6x+7y=6$,

7. Find the resultant of forces 4P and 3P parrallel to OX and OY, acting at the point (a, a) and a couple in the plane XOY whose moment is $+3a$P.

Find the intercept made by its line of action on OY.

[*Ans.* 5P at $\tan^{-1}$ 3/4 with OX ; $\frac{1}{2}a$.

8. The sides of a square ABCD are 3 in. long. Forces 5, 1, 3, 2, 3/2 act from A to B, B to C, C to D, A to D, A to C respectively. Prove that this system of forces is not equivalent to a couple and ffnd the point where the line of action of its resultant meets BC.

[*Ans.* $1\frac{1}{5}$ in. from B towards C.]

9. A system of forces acting on a rigid lamina in its plane is reduced, firstly to a single force acting at a point A in the lamina, and secondly to a single force acting at another point B and a couple of moment G. Prove, that if the system

were reduced to a single force acting at the middle point of AB and a couple, the moment of the couple, would be $\frac{1}{2}$G.

10. Six coplanar forces act on a body along the sides AB. BC, CD, DE, EF and FA of a regular hexagon ABCDEFA, in which AB is one foot long ; their magnitudes are 10, 20, 30, 40, P and Q lb. respectively. Find P and Q so that the system reduces to a pure couple and show that the moment of the couple is 75 $\sqrt{3}$ lb. ft. [*Ans.* —10, 60 lb. (*P. U. 1930*)

11. Forces P, 2P respectively act along the sides AB, BC of a rectangle ABCD, and a couple of moment P(AD+2AB) acts in the plane of the rectangle in the sense ADCB. If the resultant of the system meets AB, AD in H, K respectively, find AH and AK.

[*Ans.* H is in BA produced AH$=\frac{1}{2}$ AD ; K is at D.]

12. The moments of a system of coplanar forces about the points (1, 0), (0, 2) and (2, 3) referred to rectangular axes are G_1, G_2, G_3 respectively. If θ be the inclination of the resultant to the axis of x, prove that

$$\tan\theta = \frac{G_1 - 3G_2 + 2G_3}{2G_1 - G_2 - G_3}$$

(*D.U. 1924*)

***5·77 Nature of the resultant.** (*a*) *If the algebraic sum of the moments* M_1, M_2 *about two given points A, B, in the plane of the forces be different, then the system reduces to a single force as its resultant.*

A system of coplanar forces, not in equilibrium, always reduced to either a single force or a single couple (Art. 4·61).

If the resultant be a couple, its moment about every point in the plane must be the same, which is not true in this case.

Again, since the sum of the moments of the system of forces about *any* point, in their plane is not zero (*e.g.*, about A it is M_1) the system is not in equilibrium.

Hence the system must have a single force as its resultant.

Otherwise, this can also be shown analytically as follows :—

Let A be the origin, AB the axis of X and (b, 0) the co-ordinates of B.

Then by (2) Art. 5·75, we have

$$M_2 = M_1 - bY$$

M_2 and M_1 being different, this equation shows that Y cannot vanish and thus the system cannot reduce to a single couple and cannot be in equilibrium.

(*b*) *If the algebraic sum of the moments of a system of coplanar forces about three non-collinear points* (*A*, *B*, *C*) *in their plane is the same* (M) *and not zero, the system reduces to a couple.*

In this case the system cannot reduce to a single force (R), for, if it does, its line of action must be at the same distance $\frac{M}{R}$, from the three *non-collinear* points A, B,C which is impossible.

Also the system is not in equilibrium, as the sum of its moments about *every* point in its plane is not zero.

Hence the system must reduce to a couple.

Otherwise. To solve it analytically, let A be the origin and let (x_1, y_1) and (x_2, y_2) be the co-ordinates of B and C.

Then by Art. 5·75, $M = M - x_1Y + y_1X$ *i.e.*, $x_1Y = y_1X$

Similarly $x_2Y = y_2X$

This requires that,

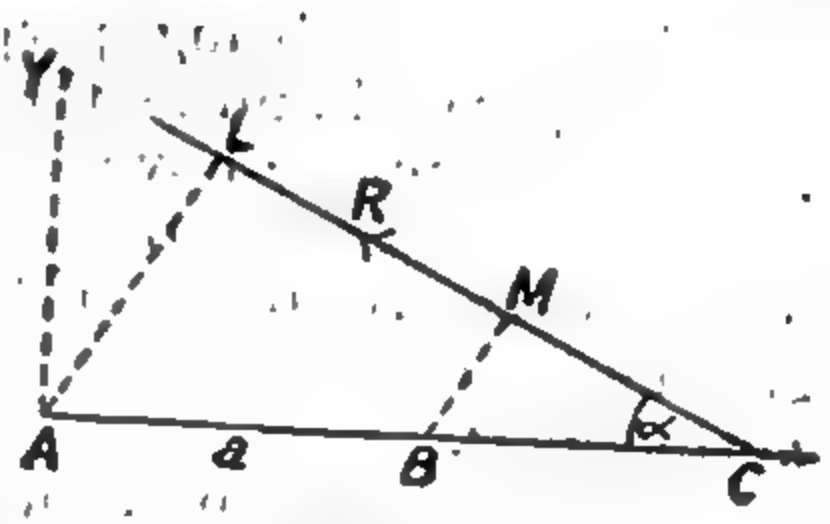
either $\frac{y_1}{x_1} = \frac{y_2}{x_2}$, *i.e.*, A, B, C are collinear which is contrary to the hypothesis,

or $X = Y = 0$ *i e.*, the resultant cannot be a single force.

Example. (*i*). *Prove that if a variable system of forces in a plane have constant moments G, G′ about two fixed points in the plane then the resultant passes through another fixed point and its minimum value is the difference of the moments dipided by the distance between the two points.*

Let A and B be the fixed points. As the moments about A and B are different, the system must reduce to a force say R, the line of action of which must interest AB, say at C, then

$$G = R.\ AL = R.\ AC \sin\alpha \quad (1)$$
and $$G' = R.\ BM = R.\ BC \sin\alpha \quad (2)$$
$$\therefore \frac{G}{G'} = \frac{AC}{BC}$$ *i.e.*, C is a fixed point on AB dividing it externally in the ratio of the given moments.



When the moments are of opposite signs, the line of action of R will divide AB internally in the ratio of the moments.

Hence all the resultant forces pass through a fixed point which divides AB internally or externally in the ratio of the constant moments.

Now from (1) and (2), we have

$$R = \frac{G}{AC \sin\alpha} = \frac{G'}{BC \sin\alpha} = \frac{G - G'}{(AC - BC)\sin\alpha} = \frac{G - G'}{AB \sin\alpha}$$

Thus R is minimum when $\sin\alpha$ is maximum $= 1$

Hence the min. value of $R = (G - G')/AB$.

Aliter. Let A (o, o) and (a, o) be the two points.

At A let the system be reduced to X, Y, G.

Then the moment about $A = G$

and the moment about $B = G - aY = G'$ (1)

G, G′ being constant Y is constant and thus only X varies.

∴ the resultant is min. when $X = 0$

∴ by (1) the min. value of $R = Y = (G - G')/a = (G - G')/AB$.

Also the equation of the line of action of the resultant is $G - xY + yX = 0$, which evidently passes through $(G/Y, 0)$, a fixed point, G and Y being constants.

Example. *Forces X, Y, Z acting at the vertices of a triangle ABC each perpendicular to BC are in equilibrium. If each is turned about its point of application in the same sense until they are all at right angles to AB, show that they form a couple, and find the moment of the couple.*

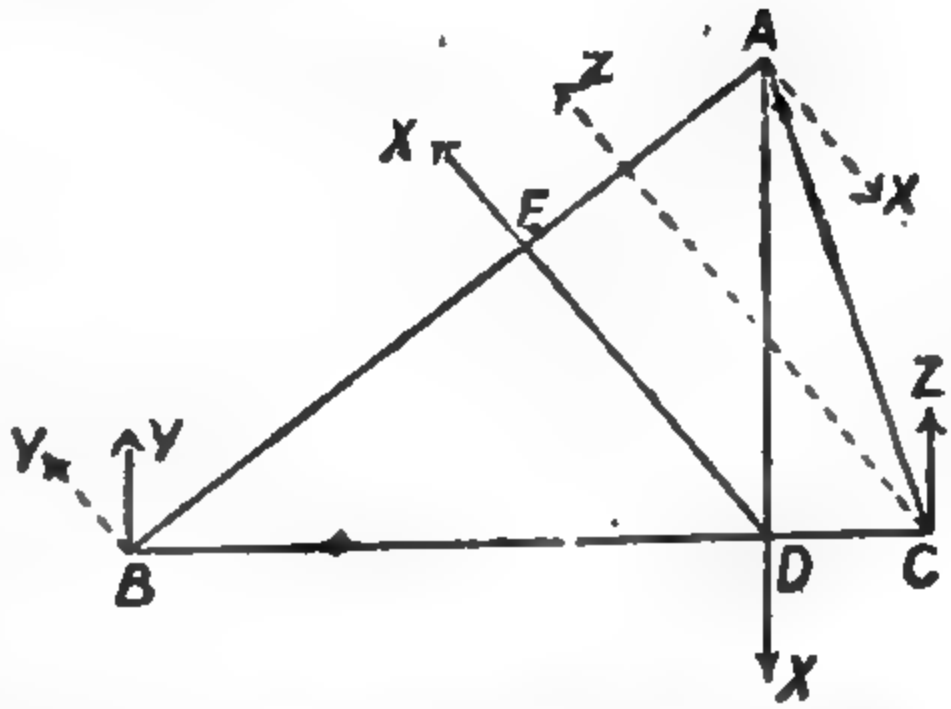
...VT. GIRLS COLL...

The forces being perp. to BC are parallel and as they are in equilibrium, we have

$$X=Y+Z$$

and that the resultant of Y and Z is equal and opposite to X, acting at D.

In the second position, the forces are shown by dotted lines. As the point of application of the resultant of parallel forces is independent of their direction, the resultant of Y and Z is a force still acting at D, equal to X acting along DE. This forms a couple with the new position of X along the dotted iine AX.



Now since AD is perp. to BC therefore $\angle EAD=90^\circ-B$

$\therefore$ the moment of the couple

$=X.AE=X.AD\cos\angle EAD=X.AD\sin B=X.c\sin B\sin B$

$=X.c\sin^2 B.$

Cor. If the forces be turned through an angle a, in the same sense, they will be equvialent to a couple of moment

$$Xc\sin a\sin B.$$

EXAMPLES XXI

1. The moments of a system of coplanar forces about three collinear points A, B, C in their plane are G_1, G_2, G_3, prove that $G_1.BC+G_2.CA+G_3.AB=0$ *(P. U. 1939)*

2. A system of forces acts on a plate in the form of an equilateral triangle of side two units. The moments of the forces about the three vertices are 10, 20 and 30 units. Show that the resultant is 10 units in magnitude and is inclined at an angle of 30° with a side of the triangle.

3. If the sum of the moments of a system of coplanar forces about any point in a given straight line in their plane is always the same, what is the nature of the resultant ?

4. Show that a system of coplanar forces can always be reduced to two parallel forces.

5. A system of forces in the plane of a triangle ABC is equivalent to a single force at A acting along the internal bisector of the angle BAC and a couple of moment G_1. If the moments of the system about B and C are G_2 and G_3, prove that

$$(b+c)G_1 = bG_2 + cG_3.$$

6. If two different systems of forces have the same moments about any three points A, B, C, the two system must have the same resultant.

Hence show that if a system of forces has moments L, M, N about any three points A, B, C and if Δ is the area of the Δ ABC, the resultant is the same as that of three forces $\frac{aL}{2\Delta}$, $\frac{bM}{2\Delta}$, $\frac{cN}{2\Delta}$ acting along BC, CA and AB.

7. Prove that in all cases a system of coplanar forces can be replaced by two forces, one of which acts through a given point and the other along a given straight line.

8. Show that a system of coplanar forces may be represented in an infinite number of ways by two forces acting through two given points in their plane.

If A and B are the fixed points and AP and BQ represent any two such forces, prove that the loci of P and Q are straight lines parallel to AB.

9. Three forces Ka, Kb, Kc act along AO, BO, CO, where O is the orthocentre of the triangle ABC. If they are rotated through the same angle θ about A, B, C, respectively, show that they become equivalent to a couple whose moment is $4K\Delta \sin\theta$, where Δ is the area of the triangle.

Rivision Questions III

1. Prove that three parallel forces two of which are equal and opposite cannot be in equilibrium.

P and Q are two parallel forces. If two equal and opposite forces S in any two parallel lines at a distance b apart in the plane of P and Q are combined with them, show that the resultant is displaced a distance $bS/(P+Q)$ or $bS/(P-Q)$ according as P and Q are like or unlike.

2. A system of co-planar forces is represented in magnitude and position by all but one of the sides of a closed polygon, taken in order.

Prove that their resultant is parallel in position and proportional in magnitude to the remaining side, and that its line of action is at a distance from that side inversely proportional to its length.

3. A door is fitted with spring hinges which normally keep it just closed. As the door is opened the hinges supply a couple of $\frac{1}{2}$ lb. ft. for every degree the door is opened. Friction resists motion with a maximum torque (couple) of 5 lb. ft. What force must be applied to the door 3 ft. from the hinges in order to keep it open 30°. [*Ans.* $6\frac{2}{3}$ lb.]

4 Four forces R, R, S, S act along the sides AB, BC, CD, DA of a rhombus. Find the sum of their moments about the centre of the rhombus and deduce that their resultant is at a distance from this point equal to

$$\tfrac{1}{2}\,BD\,(R+S)/(R-S)$$

Discuss the case when R=S.

5. Prove that if three forces k.BC. k.CA, k.AB act along the sides BC, CA, AB of a $\triangle$ ABC their resultant is a couple of moment $2k\triangle$ where $\triangle$ is the area of the triangle.

Three forces kBC, $2k$CA, $2k$AB act along the sides BC, CA, AB of a triangle ; find the single force at A and the couple to which the system is equivalent. Also find the magnitude and direction of the resultant force and the point where its line of action meets AB.

[*Ans.* kCB parallel to CB ; $2k$ $\triangle$ABC ; kBC parallel to CB, divides BA produced in the ratio 2 : 1.

6. Show that a couple and a force in its plane cannot produce equilibrium.

A uniform rod AB of weight W and length $2l$ hangs from a fixed point O by a light string OA attached to the end A, and a couple of moment N ($<lW$) is applied to the rod in a vertical plane. Find the tension in the string and the inclination of the rod to the vertical in the position of equilibrium.

If the couple is removed and a horizontal force P applied to B, find the tangents of the angles made by the string and the rod with the vertical in the equilibrium position.

[*Ans.* W, $\sin^{-1}\left(\frac{N}{W}\right)$; $\frac{P}{W}$, $\frac{2P}{W}$.]

7. Two equal unlike parallel forces, acting at fixed points A and B, form a couple of moment G ; if their lines of action are turned through a right angle, they form a couple of moment H. Show that when they both act at right angles to AB, they form a couple of moment $\sqrt{G^2+H^2}$.

8. Show that a system of forces in one plane can be reduced to a force through any given point in the plane and a couple.

There points L, M, N are takan in the sides AB, CD, EF of a regular hexagon respectively, such that LMN is an equilateral triangle and forces act along and are represented by LD, LE, MF, MA, NB and NC. Prove that the resultant will be a couple of moment $6(\triangle OLA \rightsquigarrow \triangle OLB)$ O being the centre of the hexagon. (*P. U.* 1922)

[**Hint.** Force LD at L $\equiv$ forces LD at O + couple $2\triangle LDO$
$\equiv$ LO + OD at O + couple $2\triangle LOA$
Similarly force LE at L $\equiv$ LO + OE at O + couple $2\triangle LOB$
etc.]

9. Prove that any system of co-planar forces can be represented by forces acting along the sides of a triangle.

If X_1, Y_1, Z_1 ; X_2, Y_2, Z_2, acting along the sides BC, CA, AB of the triangle ABC, represent two systems of co-planar forces, prove that the single resultants of the two systems are parallel if

$$\Sigma(Y_1Z_2-Z_1Y_2)\sin A=0$$

CHAPTER VI

CONDITIONS OF EQUILIBRIUM

6.1. Conditions of equilibrium for a system of coplanar forces acting upon a rigid body are now discussed in details.

From Art. 5·74 we deduce that the system is in equilibrium if, and only if

$$\Sigma X_1 = 0, \qquad \Sigma Y_1 = 0, \qquad \text{and } G = 0.$$

This gives us *three* conditions which can be stated as follows :—

(*a*) *The sum of the resolved parts of the forces in any chosen direction is zero.*

(*b*) *The sum of the resolved parts of the forces in any other direction* (not necessarily perpendicular to that chosen in (*a*) although it is usually most convenient so to choose) *is zero.*

(*c*) *The algebraic sum of the moments of all the forces about any arbitrary point O is zero.*

When (*a*) is satisfied, the resultant may be a force perpendicular to the direction taken or it may be a couple.

When (*a*) and (*b*) both are satisfied, the resultant cannot be a single force but may be a *pure couple*. Also if the resultant is to be a couple (*a*) and (*b*) must be satisfied.

When (*c*) is satisfied, the resultant cannot be a couple, it may reduce to a force passing through O.

When (*a*), (*b*) and (*c*) all the three conditions are satisfied, the resultant is neither a single force nor a pure couple and hence the system of forces must be in equilibrium.

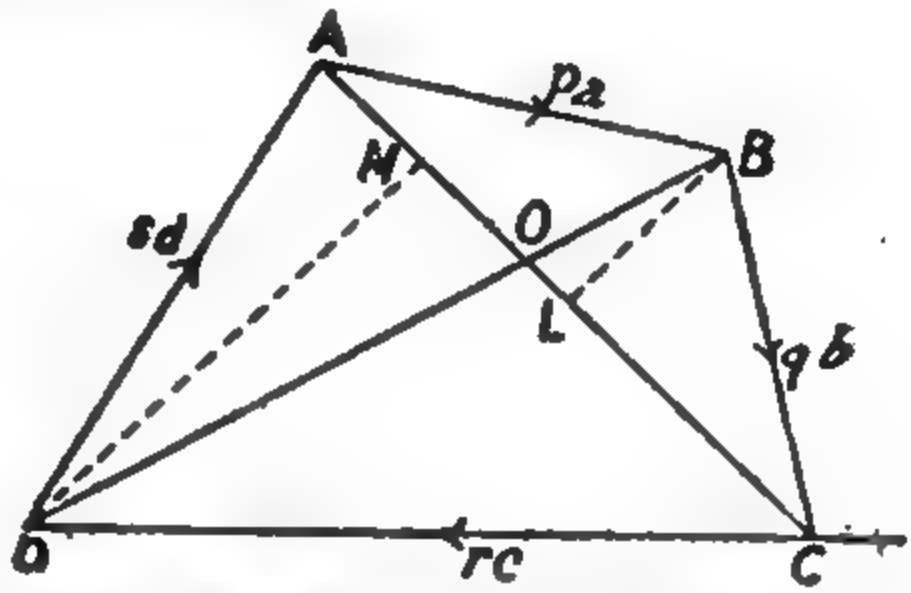
Thus (*a*), (*b*), (*c*) are the necessary and sufficient conditions for equilibrium of a system of coplanar forces.

Example. (*i*) *Forces pa, qb, rc, sd act along the sides AB, BC, CD, DA of a plane quadrilateral taken in order, a, b,*

c, d, respectively being the lengths of the sides. Prove that the forces are equivalent to a couple if

$$(p-q)\ OB=(r-s)\ OD,\ (q-r)\ OC=(s-p)\ OA$$

where O is the point of intersection of AC and BD.



Since the resultant is a couple, the sum of the resolved parts of the forces in two directions must vanish.

Resolving in a direction perp. to AC, we have

$$pa \sin \angle BAL - qb \sin \angle BCL - rc \sin \angle DCM + sd \sin \angle DAM = 0$$

or
$$p.BL - q.\ BL - r.\ DM + s.\ DM = 0.$$

$$\therefore \frac{p-q}{r-s} = \frac{DM}{BL} = \frac{OD}{OB} \text{ i. e., } (p-q)\ OB=(r-s)\ OD$$

The other result is obtained by resolving forces perp. BD.

Example. (ii) *In Ex.* (i) *if the quadrilateral is cyclic and the resultant of the forces, is a force parallel to AC, show that*

$$(p-q)\ ab=(r-s)\ cd.$$

Since the resultant is a force parallel to AC, the sum of the moments of the forces about any point in AC must be the same.

Equating the moments of the forces about A and C, we have $qb.\ AB \sin \angle ABC + rc.\ AD \sin \angle ADC$

$$= pa.\ BC \sin \angle ABC + sd.\ CD \sin \angle ADC$$

The quadrilateral being cyclic, in this case

$$\sin \angle ABC = \sin \angle ADC.$$

$$\therefore\ qb.AB + rc.\ AD = pa.\ BC + sd.\ CD$$

i.e.,
$$qba + rcd = pab + sdc.$$

Hence
$$(p-q)\ ab=(r-s)\ cd.$$

6.2. Other forms of the conditions of equilibrium.

The conditions of equilibrium may also be given in other forms such as the following :—

I. *A system of coplanar forces will be in equilibrium if the sum of the moments of the forces about each of two different*

co-planar points, say A and B, vanishes and the sum of the resolved parts of the forces in some direction not perpendicular to AB also vanishes.

The system cannot evidently reduce to a couple. It may reduce to a force, say F, then F must act along AB. But the given condition requires that the sum of the resolved parts of the forces along a line making an angle θ ($\theta \neq 90$) with AB vanishes, it means that $F \cos \theta = 0$, *i.e.*, $F = 0$. Hence the forces must be in equilibrium.

To prove this analytically, let the points A and B be (0. 0,); (a, 0). Substituting in the expression $G - xY + yX$, we have $G = 0$ and $aY = 0$ or $Y = 0$ ($\because a \neq 0$).

The third condition that the sum of the resolved parts of the forces about any line not perpendicular to AB vanishes, leads to $X = 0$.

II. *A system of coplanar forces will be in equilibrium if the sum of the moments of the forces about each of three different points, in their plane, not all lying in the same straight line, separately vanishes.*

Since the sum of the moments about one of the points. say A, vanishes, the system cannot reduce to a couple but it may reduce to a force, say F acting at A.

Again, since the sum of the moments about a second point. say B, also vanishes, the resultant force F, if it exists, must act along the line AB.

Further, since the sum of the moments about a third point, say C, vanishes, F must also pass through C, a point not lying on AB which is impossible. Hence F itself must be zero and the system must be in equilibrium.

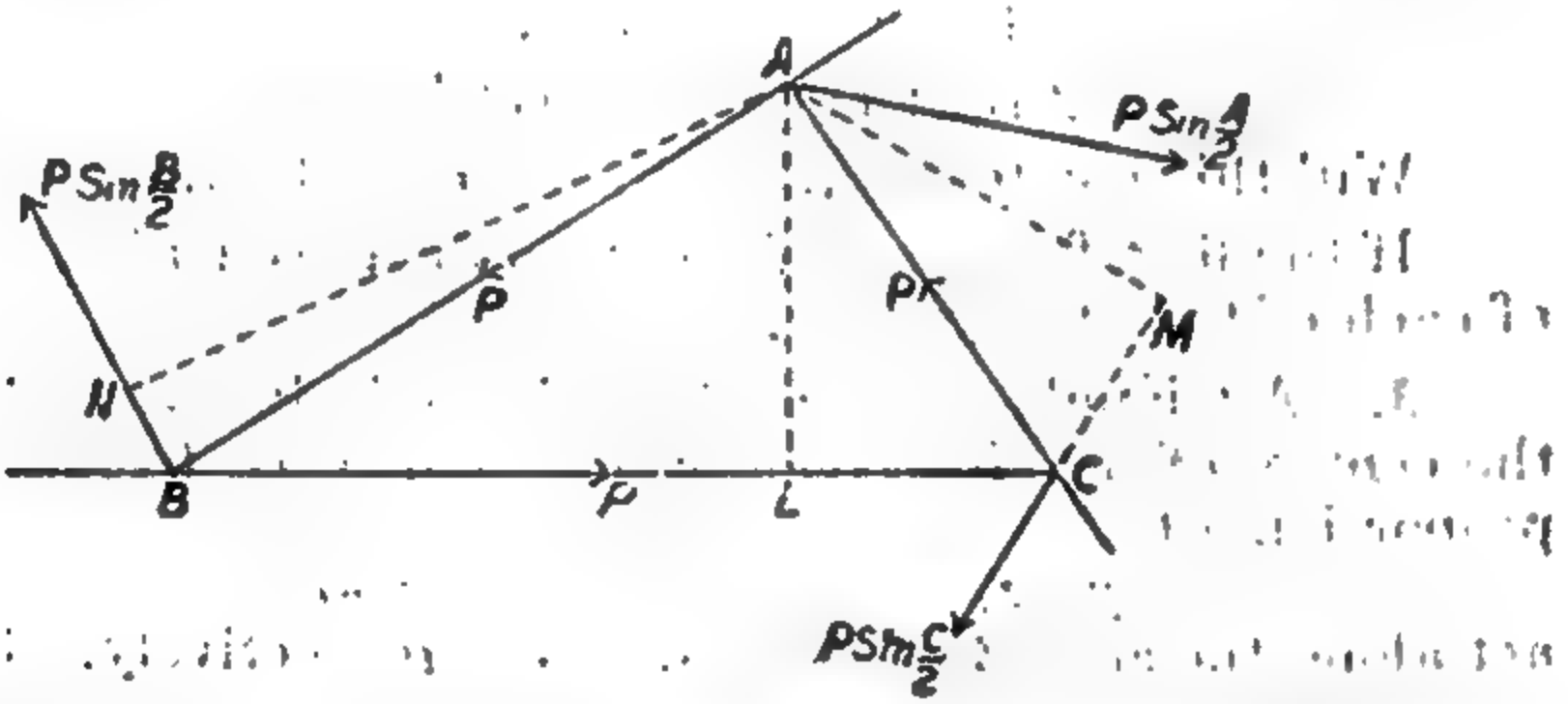
These conditions can also be proved analytically as follows :—

Let (0, 0), (x_1, y_1) and (x_2, y_2) be the three arbitrarily chosen points Substituting these points in $G - xY + yX$, an expression giving the sum of the moments of the system about any coplanar point (x, y), we have

$G = 0$, $x_1Y - y_1X = 0$ and $x_2Y - y_2X = 0$

From which we get $X=0$, $Y=0$ unless $\frac{y_1}{x_1}=\frac{y_2}{x_2}$, *i.e.*, A, B and C are collinear

Example. *Three forces each equal to* **P** *act along the sides of a triangle ABC in the same sense and forces* $P \sin\frac{A}{2}$, $P \sin\frac{B}{2}$, $P \sin\frac{C}{2}$ *act along the bisectors of the exterior angles A, B, C respectively, in the same sense opposite to that of the forces along the sides. Show that all the forces are in equilibrium.*



AL is perp. to BC; AM, AN are perp. to CM, BN, bisectors of the exterior angles C and B.

Taking moments about A, we have

$$P.AL - P \sin\frac{C}{2}.AM - P \sin\frac{B}{2}.AN$$

$$= P.b \sin C - P \sin\frac{C}{2}.b \sin\left(90°-\frac{C}{2}\right)$$

$$- P \sin\frac{B}{2}\ c \sin\left(90°-\frac{B}{2}\right)$$

$$= Pb \sin C - Pb \sin\frac{C}{2}\cos\frac{C}{2} - Pc \sin\frac{B}{2}\cos\frac{B}{2}$$

$$= Pb \sin C - \tfrac{1}{2} Pb \sin C - \tfrac{1}{2} Pc \sin B$$

$$= Pb \sin C - Pb \sin C \quad (\because b \sin C = c \sin B)$$

$$= 0$$

Similarly it can be shown that the sum of the moments about B and C vanishes. Hence the forces are in equilibrium, A, B, C being non-collinear points.

Note: This method simplifies the solution, for the forces need not now be resolved in two directions.

EXAMPLES XXII

1. State the conditions required for the equilibrium of co-planar parallel forces.

2. Forces P, Q, R, S acting in the sides AB, BC, CD, DA of a quadrilateral ABCD, are in equilibrium, show that

$$\frac{P\times R}{AB\times CD}=\frac{Q\times S}{BC\times DA}.$$

Will this condition alone ensure equilibrium ?

If the first force acts from A towards B, find the direction of each of the other forces.

3. A triangle ABC can turn freely in its own plane about the centre of its inscribed circle which is fixed and forces proportional to

$$y-z, \qquad z-x, \qquad x-y$$

act along the sides BC, CA and AB respectively. Show that the triangle remains at rest.

4. ABCD is a square whose side in 2 units in length. Forces P, Q, R, S act along two sides AB, BC, CD, DA, taken in order, and forces $F\sqrt{2}$, $T\sqrt{2}$ act along AC and DB respectively. Show that if

$$F+T=R-P, \qquad F-T=S-Q$$

the forces are equivalent to a couple of moment $P+Q+R+S$.

4 (*b*) Forces P, Q, R, S acting along the sides AB, BC, CD, DA of a quadrilateral ABCD reduce to a couple. AB and CD are parallel but not equal while AD and BC are equal but not parallel. Prove that

$$Q=S=\tfrac{1}{2}(R-P)\sec\angle ADC.$$

5. Along the sides of a regular hexagon taken in order six forces a, b, c, d, e, f are applied, show that they will be equivalent to couple if

$$e=a+b-d \quad \text{and} \quad f=c+d-a.$$

Calculate the moment of this couple in terms of a, b, c and d.

[*Ans.* $\frac{\sqrt{3}}{2} l(a+d+2b+2c)$, l=length of a side.)

6. Forces P_1, P_2, P_3, P_4, P_5. P_6 act along the sides of regular hexagon taken in order. Show that they will be in equilibrium if $\Sigma P=0$
and $P_1-P_4=P_3-P_6=P_5-P_2$. (Pub. Ser. Com.)

7. The forces P_1, P_2, P_3 act at the middle points of the sides a, b, c of the triangle ABC, and at right angles to these sides. Show that if they form a system in equilibrium with Q_1, Q_2, Q_3 acting along the same sides, then

$$Q_1.4\Delta=(P_2a-P_1b)b-(P_3a-P_1c)c$$

and two similar equations, Δ being the area of the triangle. (*B.U.*)

8. If three forces acting along and represented by the sides of the triangle ABC taken in order are in equilibrium with three forces each of magnitude X, which act along the tangents at A, B and C to the circumcircle of the triangle and whose directions are all in the same sense of rotation, prove that the triangle ABC must be equilateral and find X in terms of the length of the side. [*Ans.* X=half the side.

9. Three forces acting at the angular points of a triangle ABC along the tangents to the circumcircle, are equivalent to a couple. Show that they are in the ratio

sin 2A : sin 2B : sin 2C.

10. Forces $\sqrt{3}$ P, $\sqrt{3}$ Q, $\sqrt{3}$ R act along the altitudes AD, BE, CF and forces R—Q, P—R, Q—P along the sides BC, CA, AB respectively of an equilateral triangle ABC. Show that the system is in equilibrium.

11. Forces $2P\cos\frac{A}{2}$, $2Q\cos\frac{B}{2}$, $2R\cos\frac{C}{2}$ act along the internal bisectors of the angles A, B, C and forces R—Q, P—R, Q—P act along the sides BC, CA, AB respectively of the triangle ABC. Show that the six forces are in equilibrium.

12. Forces P, Q, R act along the sides BC, CA, AB of a triangle ABC and forces P′, Q′, R′ act along OA, OB, OC where

O is the centre of the circumscribed circle. Prove that if the six forces are in equilibrium,

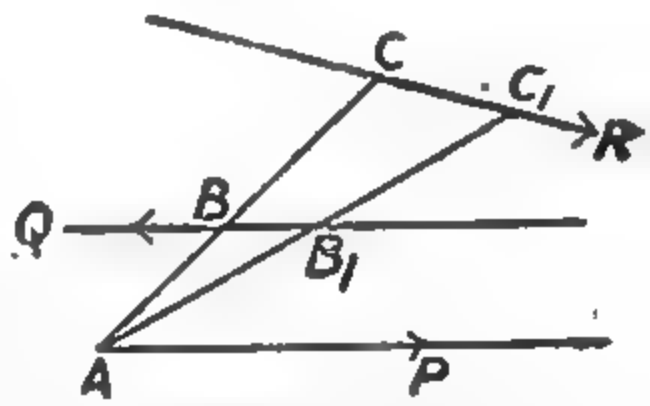
$$P \cos A + Q \cos B + R \cos C = 0$$

and

$$\frac{PP'}{a} + \frac{QQ'}{b} + \frac{RR'}{c} = 0$$

***6.3. Three forces acting on a rigid body.** *If three forces acting on a rigid body keep it in equilibrium, they must be coplanar.*

Let P, Q, R be the three forces, A any point on the line of action of P and B a point on that of Q.



As P and Q meet the line AB, they separately can have no tendency to turn the body about AB, and since P, Q, R in are equilibriuu, they together can have no effect to turn the body about AB, so that the third force R cannot turn the body about AB.

Hence AB must meet the force R in some point, say C.

Similarly if B_1 is another point on Q, AB_1 must meet R, say in C_1, so that the line of action of R must lie in a plane ABB_1 or in other words the lines of action of *Q and R are coplanar and they lie in a plane* which passes through A.

Similarly by taking another point A_1 on P, it can be proved that Q and R are coplanar and that their plane meets the line of action of P in A_1 so that the plane in which Q and R lie, contains two points A and A_1 on the line of action of P *i.e.*, it contains the whole line of action of P.

Hence the three forces are coplanar.

6 31. Conditions of equilibrium of three forces acting on a rigid body.

From the above (Art 6·3) and articles 3.3 and 4.11, (Cor.), it follows that :—

If three forces acting upon a rigid body are in equilibrium :—

(i) *they must be coplanar.*

(ii) *they must either meet at a point or be parallel.*

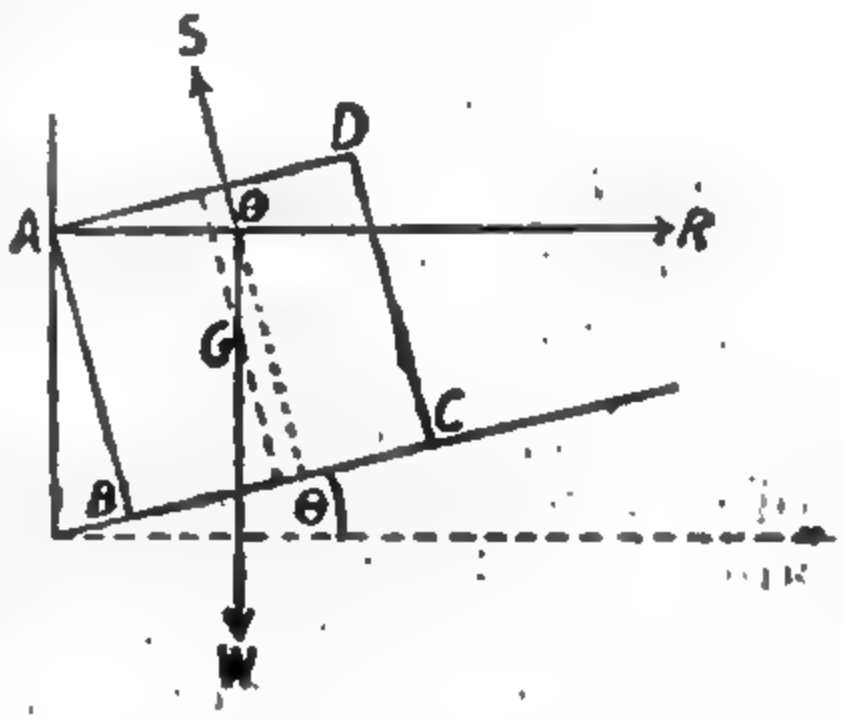
(iii) (a) *when concurrent, each is proportional to the sine of the angle between the other two.*

(b) *when parallel, each is proportional to the distance between the other two.*

EXAMPLES XXIII

1. *A uniform cubical block of weight W rests on a smooth inclined plane with one of its edges horizontal and in contact with a smooth vertical wall. The inclination of the plane is θ. Find the pressures on the plane and on the wall.*

ABCD is the section of the block made by a vertical plane through its centre of gravity G, meeting the inclined plane in BC and the wall in A.



The forces acting on the block are :—

(i) the resultant reaction R. of the wall acting at A along AO perp. to the wall, the wall being smooth.

(ii) the weight W acting along a vertical line through G.

(iii) the resultant reaction S of the inclined plane perp. to BC, the plane being smooth.

The block being in equilibrium under three forces, the forces must lie in the same plane ABCD, and must meet at a point, so that the reaction S of the inclined plane is a force perp. to BC and passing through O, the point of intersection of R and W. (*Note that S. does not pass through G*).

Now by Lami's theorem, we have

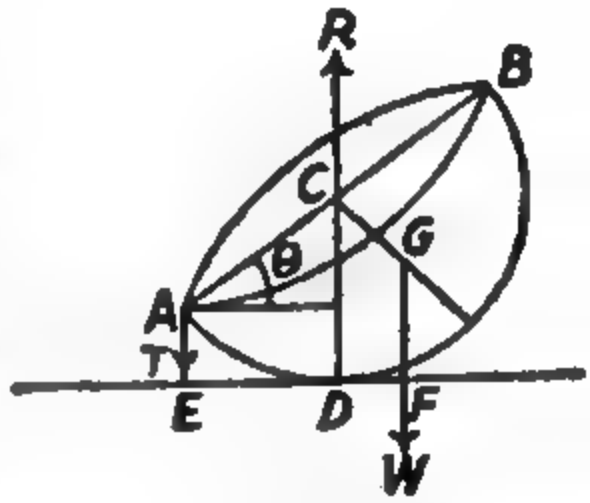
$$\frac{R}{\sin \angle SOW} = \frac{S}{\sin \angle WOR} = \frac{W}{\sin \angle SOR}$$

$$\frac{R}{\sin (180° - \theta)} = \frac{S}{\sin 90°} = \frac{W}{\sin (90° + \theta)}$$

$$i.e. \qquad \frac{R}{\sin\theta} = \frac{S}{1} = \frac{W}{\cos\theta}$$

Hence $R = W \tan\theta$ and $S = W \sec\theta$.

2. *A hemisphere of radius a and weight W is placed with its curved surface on a smooth table and a string of length $l(<r)$ is attached to a point on its rim and to a point on the table. Find the position of equilibrium and prove that the tension of the string is* $\frac{3}{8} W \frac{a-l}{\sqrt{2al-l^2}}$.



Let the plane face of the hemisphere be inclined at an angles θ to the horizontal since the hemisphere is in equilibrium under only three forces *viz.*, its weight, the reaction of the table and the tension of the string, the forces must lie in the same vertical plane.

Also, since the table is smooth, the reaction at the point of contact D must be normal to both, the surface of the table and that of the sphere, so that its line of action DC must be vertical. The weight of the hemisphere acting at G($CG = \frac{3}{8}a$) being also a vertical force, the third force, the tension T must also be vertical.

Thus the sphere will adjust itself such that the string EA is vertical and lies in the vertical plane through C.

Now $DC - EA = a - l = a \sin\theta$.

$$\therefore \sin\theta = \frac{a-l}{a} \qquad \text{or} \qquad \tan\theta = \frac{a-l}{\sqrt{2al-l^2}}$$

Making moments about D, we have

$$T.ED = W.DF$$

$$i.e., \quad T.\, a\cos\theta = W.CG \sin\theta = W.\tfrac{3}{8}a \sin\theta,$$

$$\therefore \qquad T = \tfrac{3}{8} W \tan\theta = \frac{\frac{3}{8}W\,(a-l)}{\sqrt{2al-l^2}}$$

2. A uniform solide hemisphere of weight W has a particle of the same weight attached to its rim. The hemisphere is

suspended from the centre of its plane face. Prove that in the position of equilibrium, the axis of the hemisphere is inclined at $\tan^{-1}(\frac{8}{3})$ to the vertical.

3. A sphere of radius a, and weight W, rests on a smooth inclined plane supported by a string of length l with one end attached to a point on the surface of the sphere and the other end fastened to a point on the plane. If the angle of incl nation of the plane to the horizontal is α, show that the tension of the string is

$$W(a+l)\sin\alpha/\sqrt{l^2+2al}$$

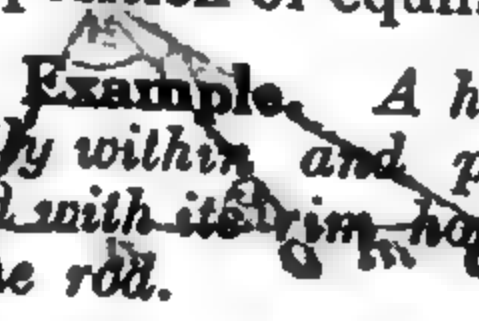
4. A board of weight W in the shape of a regular polygon of n sides, rests with one corner A against a smooth vertical wall, the adjacent corner B being attached to the wall by a string whose length is equal to a side of the polygon. Show that the angle θ which AB makes with the wall is

$$\tan^{-1}\left(\frac{1}{3}\cot\frac{\pi}{n}\right)$$

and that the pressure at A is $\frac{W}{3}\cot\frac{\pi}{n}$

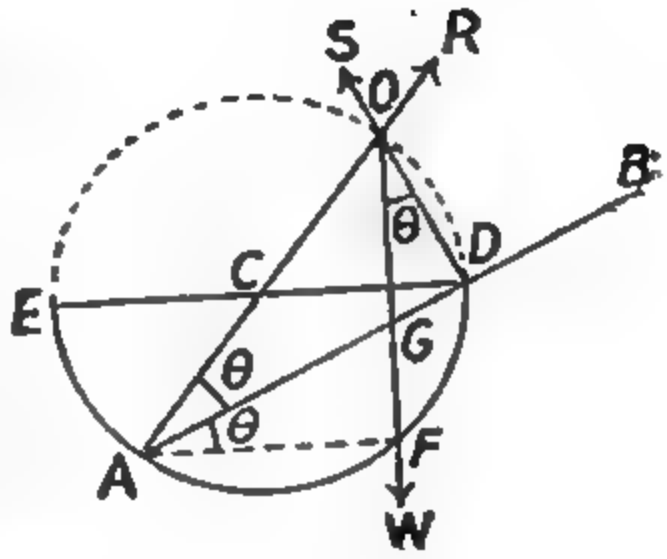
5. A picture frame, rectangular in shape, rests against a smooth vertical wall ; it is suspended from two points in the wall by parallel strings attached to two points in the upper edge of the back of the frame, the length of each string being equal to the height of the frame. Show that if the centre of gravity of the frame coincide with the centre of figure, the picture will hang against the wall at an angle $\tan^{-1}(b/3a)$ to the vertical, where a is the height and b is the thickness of the picture.

6.4 Sometimes a mere geometrical consideration gives the position of equilibrium,



Example. *A heavy uniform smooth rod of length 2l, rests partly within and partly outside a* **smooth** *hemispherical bowl fixed with its rim horizontal. Find the position of equilibrium of the rod.*

DEF is a section of the bowl by a vertical plane through the centre C.



In the position of equilibrium, let the rod AB be inclined at an angle θ to the horizontal.

The bowl and the rod both being smooth, the reaction at D where the rod rests on the rim, must act along DO perpendicular to the rod at D. Aiso the reaction at A acts along AO, the normal at A to the surface of the bowl.

$\angle$ADO being a right angle AO and DO must meet on the circumference of the circle DEF.

Similarly the vertical through O and the horizontal line through A must meet at F a point on the circumference.

The only other force being the weight of the rod, the three forces must meet, *i. e.*, the vertical from O must bisect the rod at G,

Now $\theta = \angle GAF = \angle ADC = \angle CAD$, so that $\angle OAF = 2\theta$.

Since $AO = 2a$ and $AG = l$, we have

$AF = AO \cos 2\theta$ and also $= AG \cos \theta$

Hence $$2a \cos 2\theta = l \cos \theta,$$

which gives θ, the position of equilibrium of the rod.

To find the reactions R and S, we apply Lami's Theorem,

and get $$\frac{R}{\sin \theta} = \frac{S}{\cos 2\theta} = \frac{W}{\cos \theta}.$$

6·41. The following trigonometrical formulae sometimes prove very useful for finding the position of equilibrium of bodies.

If D be any point in the base BC of a triangle ABC, $BD = m$, $DC = n$,

$\angle BAD = \alpha$, $\angle DAC = \beta$, then the angle $\theta = \angle ADC$ is given by

$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$ .....(1)

$(m+n)\cot\theta = n\cot B - m\cot C$ .....(2)

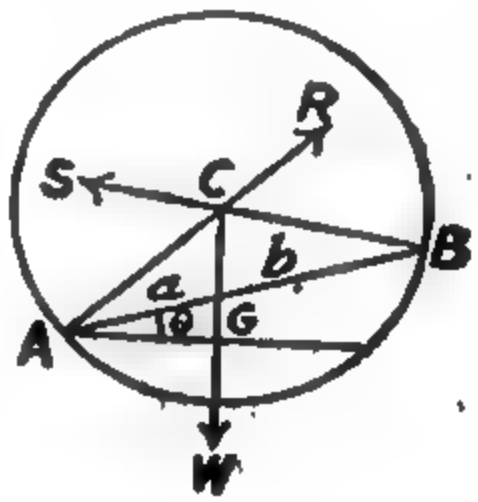
In particular if D bisects BC,

$2\cot\theta = \cot\alpha - \cot\beta$ and $2\cot\theta = \cot B - \cot C$.

The student is advised to prove and remember these results.

6·42. Example. *A heavy rod whose centre of gravity divides it into two parts a and b is placed within a smooth sphere. The rod subtends an angle 2α at the centre. Find its position of equilibrium.*

The surface of the sphere being smooth, the reactions R and S at the ends A and B of the rod must act along the radii AC and BC. The only other force being its weight, the vertical line from C must pass through G such that $AG = a$ and $GB = b$.



Since the rod subtends an angle 2α at the centre.

$\therefore \angle CAB = \angle CBA = 90° - \alpha$.

Let θ be the angle which the rod makes with the horizontal $\angle CGB = 90° - \theta$.

Using formula (2) Art 6·41 we have

$(a+b)\cot\angle CGB = b\cot\angle CAB - a\cot\angle CBA$.

or $(a+b)\tan\theta = b\tan\alpha - a\tan\alpha$

$\therefore$ the position of equilibrium is given by

$(a+b)\tan\theta = (b-a)\tan\alpha$.

Cor. *If the rods were uniform i.e., $a=b$, $\theta=0$, i.e., a uniform rod cannot rest in a position other than the horizontal unless the surface of the bowl is rough instead of smooth.*

Note. To find R and S, we take moments about A and B.

$S(a+b)\sin(90°-\alpha) = W a\cos\theta$ i.e., $S = W\dfrac{a\cos\theta}{(a+b)\cos\alpha}$.

and $R(a+b)\sin(90°-\alpha) = Wb\cos\theta$ i.e., $R = W\dfrac{b\cos\theta}{(a+b)\cos\alpha}$.

EXAMPLES XXIV

1. A rod whose centre of gravity divides it into two portions whose lengths are a and b $(b>a)$, has a string tied to its ends and the string is slung over a smooth peg. Show that the angle θ which the rod makes with the horizon is given by

$$(a+b)\tan\theta=(b-a)\cot\alpha$$

where 2α is the angle subtended at the peg in the position of equilibrium.

2. A beam which is divided by its C. G. into two portions of lengths a and b rests in equilibrium on two smooth planes inclined at angles α, β respectively to the horizon, the planes intersecting in a horizontal line. Show that θ the inclination of the beam to the horizon is given

$$(a+b)\tan\theta=b\cot\beta-a\cot\alpha.$$

3. A triangular lamina ABC is suspended from a point O by light strings fastened to the points A,B and hangs so that the side BC is vertical. Prove that, if α, β are the angles which the strings AO, BO make with the vertical, then

$$2\cot\alpha-\cot\beta=3\cot B. \qquad (Aligarh\ 1943)$$

4. A uniform rod AB is in equilibrium at an angle α with the horizontal with its upper end A resting against a smooth peg and its lower end B attached to a light cord which is fastened to a point C in the same level as A. Prove that the angle β at which the cord is inclined to the horizontal is given by the equation : $\tan\beta=2\tan\alpha+\cot\alpha.$
and show that $AC(1+2\tan^2\alpha)=AB\sec\alpha.$ (D.U. 1940)

5. Inside a smooth hollow sphere is placed a rod of weight W equal in length to the radius of the sphere, the C.G. of the rod being at a point one third of its length from one end. Find the position of equilibrium of the rod and also the pressures at the two ends.

[*Ans.* At $\tan^{-1}(3\sqrt{3})$ with vertical ; $2W/\sqrt{7}$; $W/\sqrt{7}$.)

6. A uniform heavy rod, whose length is equal to the diameter of a smooth hemispherical bowl which is fixed with its axis vertical, rests with one end in the bowl and the other

end outside the bowl. Prove that the inclination of the rod to the horizontal is about 30° 32′.

7. A bowl is formed from a hollow sphere of radius a; it is so fixed that the radius of the sphere drawn to each point in the rim makes an angle α with the vertical, and the radius drawn to a point A of the bowl makes an angle β with the vertical. If a smooth uniform rod remains at rest when placed with one extremity at A, and with a point in its length on the rim of the bowl, show that the length of the rod is

$$4a \sin \beta \sec \tfrac{1}{2}(\alpha-\beta),$$ (P.U. 1932)

8. Show that the greatest inclination to the horizon at which a uniform rod can rest partly within and partly without a fixed smooth hemispherical bowl is $\sin^{-1}\frac{1}{\sqrt{3}}$. (D.U. 1934)

9. A light rod rests wholly within a smooth hemispherical bowl of radius r, and a weight W is clamped on the rod at a point whose distances from the ends are a and b. Show that θ, the inclination of the rod to the horizon in the position of equilibrium is given by the equation

$$2\sqrt{r^2-ab} \sin \theta = a-b.$$

10. One end of a string of length l is attached to one end A of a smooth uniform rod of length a, and the other end of the string is attached to a light ring C which slides on the rod. The string passes over a small smooth pulley O, prove that in the position of equilibrium the inclination θ of the rod to the horizontal is given by $a \cos^3 \theta = l \sin \theta$

11. A string 16 ft. long has one end attached to the entremity of a smooth uniform heavy rod of length 10 ft. and at the other end carries a weightless ring, which slides on the rod. The rod is suspended by means of the string from a smooth peg; show that the angle θ which the rod makes with the horizon is given by $\tan \theta = \frac{1}{2}$

CHAPTER VII

THE SOLUT.ON OF PROBLEMS

7·1. The set of conditions of equilibrium as given in Art. 6·1 is the most useful for solution of problems involving equilibrium of a rigid body under co-planar forces. This leads to the following procedure :—

Equate to zero the sum of the resolved parts of the forces along two suitable perpendicular lines.

Equate to zero the sum of the moments of the forces about some suitable point.

In addition to the three equations so obtained, geometrical relation, holding between the bodies under consideration, may also be written down.

All these will generally suffice to determine the positions of equilibrium and the magnitudes and directions of the unknown forces.

The procedure can often be much simplified by the proper choice of the directions along which the forces are resolved. In general, *horizontal* and the *vertical* directions are the most suitable In problems involving an inclined plane the directions along and perpendicular to the plane may be found convenient.

The point about which moments are to be taken should be one which lies on the lines of action of as many forces as possible, so that as few of the forces as possible are introduced into the equation of moment.

In case a body is hinged to a point, moment is generally taken about the hinged point so as to avoid the unknown reaction at the hinge.

If a body is hinged along a line, moment is taken about the line joining the himges.

Conditions of equilibrium are also satisfied if sum of the moments about three non-collinear points, in the plane of the forces, be equated to zero.

This is sometimes more convenient, the points being suitably selected.

72. We now give a set of easy examples on the direct application of the procedure indicated above.

EXAMPLES XXV

1. *A uniform rod of weight W rests with its one end on a smooth inclined plane AB, and the other end against a smooth vertical wall BD, B being higher than A. A string is attached to the lower end of the rod and passing over the pulley at B supports a weight P. If θ be the inclination of the rod to the vertical and α the inclination of the plane show that*

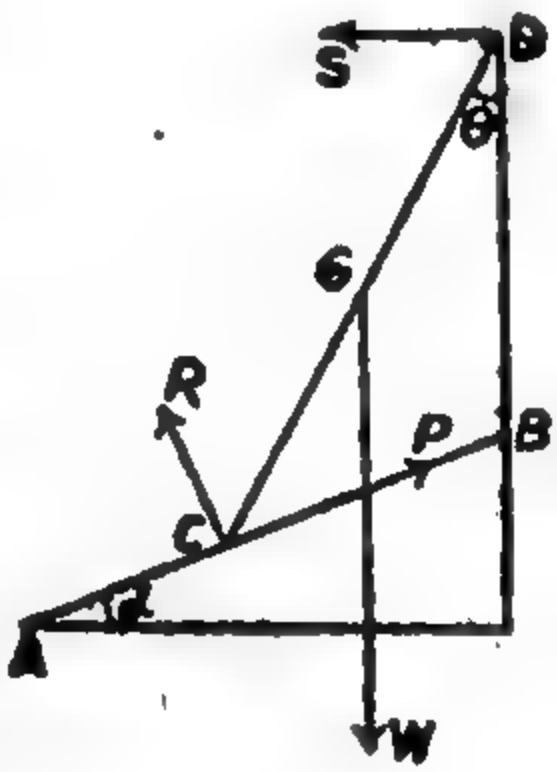
$$W \tan\theta = 2 \sec\alpha\,(P - W \sin\alpha).$$

The forces acting on the rod CD ($=2a$) are :—

S, the reaction of the wall, perpendicular to the wall at D ; R, the reaction of the inclined plane, perp. to the plane at C.

P, the tension of the string acting along CB.

W, the weight of the rod acting vertically at its middle point.



Taking moments about C, (through which two forces R and P pass), we have

$W \cdot a \sin\theta = S\ 2a \cos\theta$ *i.e.*,

$$S = \tfrac{1}{2} W \tan\theta \quad \ldots\ldots\ldots(1)$$

Resolving along the plane CB, we have

$$P = W \sin\alpha + S \cos\alpha \quad \ldots\ldots(2)$$

From (1) and (2), we get

$$P = W \sin\alpha + \tfrac{1}{2} W \tan\theta \cos\alpha$$

i.e., $W \tan\theta = 2 \sec\alpha\,(P - W \sin\alpha)$

Note. If the reaction R were required, one more equation could be written down by resolving forces in a direction perpendicular to the plane BC.

Note 2. If the plane were rough, the rod may rest in this position without the help of the string. In that case P will represent the force of friction and will be given by (2)

2. A uniform beam AB of weight W rests on a smooth floor at A and on a smooth inclined plane of inclination α at B. The beam is maintained in a position inclined at θ to the horizontal by a string fastened to B which passes over a smooth pulley at the top of the plane, and supports a weight P hanging freely. Show that the beam will rest in all positions if

$$2P = W \sin \alpha.$$

3. A uniform rod AB of weight 20 lb. and of length 3 ft. is freely hinged at one end A and a weight of 20 lb. is hung from the other end B; the rod is supported in a horizontal position by a string BC attached to a fixed point C 15 in. vertically above A. Prove that the reaction at A is inclined to the horizon at an angle $\tan^{-1}(\frac{5}{3\,6})$ (*P. U.* 1925)

4. A uniform rod AB of weight W is smoothly hinged to a fixed support at A and weight 2W is suspended from B; the rod is held horizontally by a string from B attached to a point C vertically above A. If AB=AC, find the magnitude of the reaction at A and show that its line of action makes $\tan^{-1}(\frac{1}{2})$ with the horizontal. [*Ans.* 2·55 W]

5. A uniform bar AB of weight W free to turn about A, has a small smooth light ring fixed to it at B. A light string attached to a point C on a level with A and such that AC=AB, passes through the ring and carries a particle of weight W/10 at the other end. Show that in the equilibrium position the bar is inclined to the vertical at an angle $\sin^{-1}1/8$.

6. A beam AB of weight W is divided by its centre of gravity C into two portions AC and CB, whose lengths are a and b respectively. The beam rests in a vertical plane on a smooth floor AD and against a smooth vertical wall DB. A

string is attached to a hook at D and the beam at a point P. If T be the tension of the string and θ and ϕ be the inclination of the beam and the string respectively to the horizon, show that

$$T=W\frac{a\cos\theta}{(a+b)\sin(\theta-\phi)} \quad (P.\ U.\ Supp.\ 1941)$$

7. A circular disc BCD of radius a and weight W is supported by a band of inappreciable weight and thickness which surrounds the disc along the arc BCD and is fastened at its extremities to the point A in a vertical wall, the portion AD touching the wall and the plane of the disc being at right angles to the wall. If the length of the band not in contact with the disc be $2l$, show that the tension of the band is

$$\frac{W}{2}\cdot\frac{a^2+l^2}{l^2}$$

(*D. U.* 1940)

8. A thin ring of radius R and weight W is placed round a vertical cylinder of radius r and is prevented from falling by a nail projecting horizontally from the cylinder. Show that the horizontal pressure between the cylinder and the ring is $Wr/2\sqrt{R^2-r^2}$.

9. A heavy rectangular board ABCD of uniform thickness and of weight W is supported at the middle point of AB, about which it can turn freely in a vertical plane. Weights P, Q (P>Q) are attached by strings to the corners B, D and hung freely. Prove that in the position of equilibrium, the inclination of AB to the horizon will be

$$\tan^{-1}\frac{a(P-Q)}{b(W+2Q)},$$

where a and b are the lengths of the sides AB, AD of the rectangle.

10. A rectangular gate of weight 40 lb. is 8 ft. wide and 4 ft. high. It is supported upon two hinges one 6 in. below the top and the other 6 in. above the bottom of one of the vertical sides. If all the weight of the gate is taken by the lower hinge, find the reaction at the upper hinge and the vertical and horizontal reactions of the lower hinge.

[*Ans.* $53\frac{1}{3}$ lb. wt.; 40 lb.; $53\frac{1}{3}$ lb. wt.]

11. A smooth hemispherical bowl, of diameter a is placed, so that its edge touches a vertical wall, a uniform rod AB of weight W is in equilibrium inclined t 60° to the horizon with one end A resting on the inner surface of the bowl and the other end B resting against the wall, show that the length of the rod must be $a+a/\sqrt{13}$.

Find the reactions against the wall and the bowl.

[*Ans.* $W/2\sqrt{3}$; $\sqrt{13}W/2\sqrt{3}$.]

15. A smooth tube of radius r has hemispherical end. The tube is fixed vertically with the hemisdherical end at the bottom. A rod of length l is placed entirely in the tube and assumes a position of equilibrium, being inclined at an angle θ to the horizontal. If α is the angle between the reaction at the lower end and the horizontal, show that

$$l\cos\theta=r(1+\cos\alpha) \quad \text{and} \quad \tan\alpha=2\tan\theta.$$

Prove that the centre of the rod is level with the centre of the hemisphere when $r=\sqrt{3}\,l/4$.

13. A uniform square plate ABCD of weight W is kept in equilibrium in a vertical plane with A resting on a smooth horizontal floor and B against a smooth vertical wall perpendicular to the plate by means of a force P which acts parallel to AB at a point in the side AD. If AB is inclined to the horizontal at an angle of 60°, show that the least value of P is $\frac{1}{13}$ W $(3\sqrt{3}-1)$.

***7.3. Equilibrium of Connected Bodies.** When two or more bodies are in contact, the equilibrium of each is considered separately, taking into account all the forces which arise in virtue of its connection with the others, in addition to the special forces acting upon it.

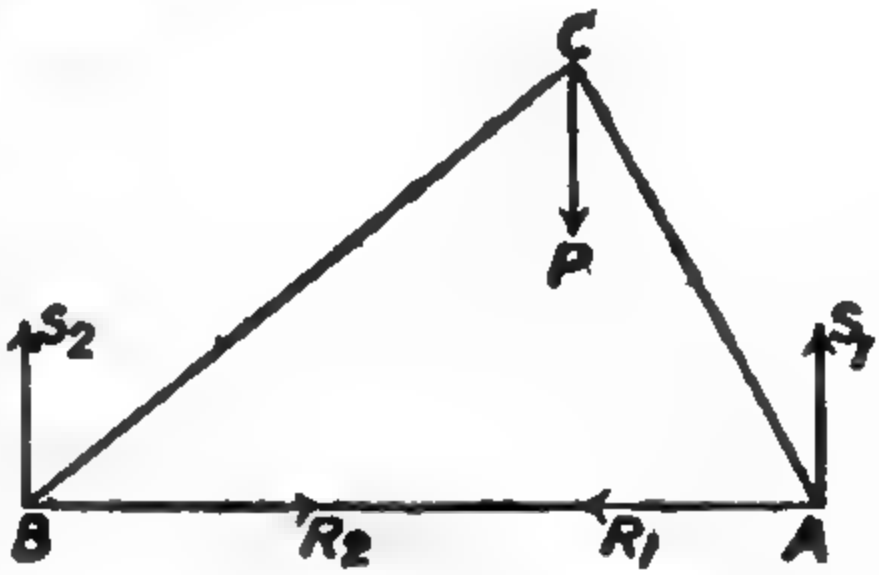
In some cases, however, it is found more useful to write down the equations of equilibrium by taking two bodies together regarding them as a single body. This avoids the appearance in the equations of equilibrium, of the unknown reactions between the bodies.

Example (*i*) *Two light rods AC, CB are freely jointed at C. They stand in a vertical plane with the ends A and B on a smooth*

horizontal table pressing against two fixed obstacles. Prove that if a vertical thrust P is applied at C, the horizontal thrust against either obstacle is P cos A cos B/sin C.

If the lengths of the rods be varied, prove that when the angle C is given, the thrust is a maximum when AC=CB.

Let AC, BC be the rods, R_1, R_2 the reactions of the obstacles and S_1, S_2 the vertical reactions of the table, on the rods.



To avoid the action at C, we resolve horizontally and vertically for the *whole system*, we have,

$$R_1 - R_2 = 0 \text{ } i.e., \text{ } R_1 = R_2 \ldots\ldots\ldots \quad (1)$$

which shows that the reaction of the obstacle on either rod is the same,

and $$P = S_1 + S_2 \ldots\ldots\ldots (2)$$

Now, taking moments about C for one rod AC, we have

$R_1 . AC \sin A = S_1 . AC \cos A$ *i.e.*, $S_1 = R_1 \tan A$,

Similarly for the rod BC, $S_2 = R_2 \tan B = R_1 \tan B$.

$$\therefore \quad P = S_1 + S_2 = R_1 (\tan A + \tan B)$$

$$= R_1 \frac{\sin (A+B)}{\cos A \cos B}$$

$$= R_1 \frac{\sin C}{\cos A \cos B}$$

i.e., $$R_1 = R_2 = P \cos A \cos B / \sin C.$$

Now, since $\cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)]$

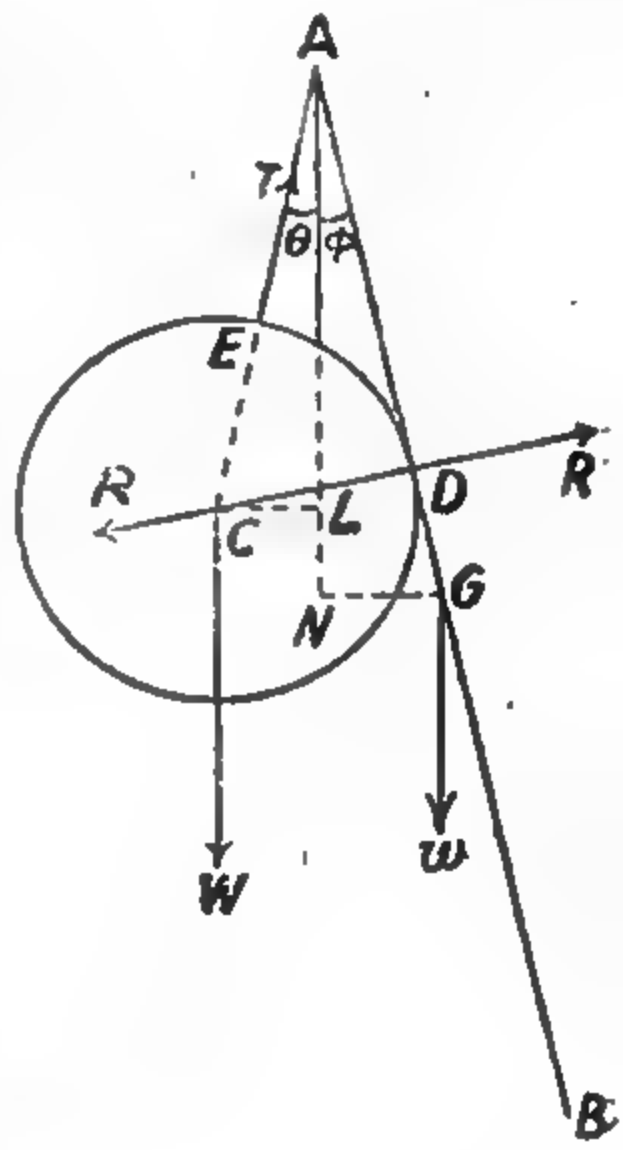
$= \frac{1}{2} [\cos(A-B) - \cos C]$,

and C being given, R_1 or R_2, will be a maximum when A=B *i.e.*, AC=BC.

Example (*ii*) *A syhere of weight W and radius r is hung by a string of length l, and a uniform rod of weight w and length 2a has one end attached to the same point and can freely turn about it.*

If the rod rests touching the sphere, show that the angle θ made by the string with the vertical is given by the equation.

$$\tan\theta = w\,a\cos^2 a/[Wr + w\,a\cos a\sin a]$$
where $\sec a = (l+r)/r$ *(B. U)*



The rod AB and the sphere with centre C touch each other at D where the action and reaction are equal, each equal to say R.

For the sphere, since R and W meet at C, the tension of the string EA, which is the only other force on it must also pass through C *i e*, CEA must be a straight line and $CA = CE + EA = r + l$.

The angle which the string makes with the vertical $= \angle CAL = \theta$.

Let $\angle LAD$, the angle which the rod makes with the vertical $= \phi$

Then $\cos \angle ACD = \sin(\theta+\phi)$

$$= \frac{CD}{AC} = \frac{r}{r+l} = \cos a \text{ (given)}$$

Hence $\angle ACD = a$ and $\phi = \angle CAD - \theta$

$$= 90 - a - \theta.$$

Taking moments about A for the whole system, we have

$W.CL = w.NG$ *i.e.*, $W(l+r)\sin\theta = wa\sin\phi$

Now to express the result in the given form we substitute, the values of $l+r$ and ϕ and get

$W r \sec a \sin\theta = wa\cos(a+\theta) = wa(\cos a\cos\theta - \sin a\sin\theta)$

i.e., $\tan\theta = wa\cos^2 a/[Wr + wa\sin a\cos a]$.

EXAMPLES XXVI

1. A sphere of weight W radius a hangs by a string of length b from a point A and a weight W′ is also suspended by a string from the point A, the string being long enough for the weight W′ to hang below the sphere, show that the first string is inclined to the vertical at an angle θ, where

$$(W+W')(a+b)\sin\theta = a\,W'.$$

2. Two equal uniform rods of length $8a$ are freely jointed and rest symmetrically over two smooth pegs fixed in a horizontal line at a distance a apart. Show that each rod is inclined to the vertical at an angle of 30°

3. Two smooth uniform rods OB, OC each of length l and weight W are smoothly jointed at O, and are placed symmetrically on a fixed horizontal rod of length $2a$, so as to form a letter A. A light vertical string connects the joint O to the mid-point of the fixed rod. When the rods OB, OC are inclined at θ to the horizontal, show that the tension on T the string is given by

$$T = W\left(\frac{l}{a}\cos^3\theta - 2\right).$$

4. Two equal uniform heavy rods, each of length $2l$ are smoothly jointed together at an end and are placed so as to rest symmetrically over a smooth cylinder of radius a fixed with its axis horizontal. Show that in the position of equilibrium the angle θ between each rod and the vertical is given by

$$l\sin^3\theta = a\cos\theta.$$

5. Two equal beams AB, AC each of weight W connected by a hinge at A, are placed in a vertical plane with their extremities B and C resting on a smooth horizontal plane. They are kept from falling by strings connecting B and C with the mid-paints of the opposite beam. Show that the tension in each string is

$$\tfrac{1}{8}W\sqrt{9\cot^2\theta + 1}.$$

θ being the inclination of each beam to the horizon.

6. Two uniform bars AB, AC each of length $2l$ and weight W are freely jointed at A and stand in equilibrium in a vertical plane and their mid-points are jointed by a light elastic string of natural length $2a$ and modulus λ. If 2θ be the angle between the rods, show that θ satisfies the equation

$$\frac{1}{\sin\theta} + \frac{W}{\lambda\cos\theta} = \frac{l}{a}.$$

7. A uniform rod AB 6 ft. long weighing 15 lb. is smoothly hinged at B to another uniform rod BC, 8 ft. long weighing 20 lb. The ends A, C are fastened to smooth hinges in a horizon-

tal line 10 ft. apart with B below AC, and a horizontal force 30 lb. wt. is applied to the rod BC along the straight line joining the middle points of BC and AB in the same sense. Find the horizontal and vertical components of the reaction at the hinge B.

[*Ans.* 3 lb., $3\frac{1}{2}$ lb.]

8. A hollow bottomless cylinder of mass M stands upright on the ground. Two equal smooth spheres are placed within it, one resting on the ground and the other supported by the cylinder and the other sphere. If the mass of either sphere be m, its radius a, and the radius of the cylinder $\frac{3}{2}a$, show that the cylinder is on the point of toppling over provided that $2m=3M$.

9. A cylindrical log of weight W and radius $4r$ lies on a smooth floor with its curved surface in contact with a smooth vertical wall A second cylindrical log of radius r and weight w lies on the floor besides the first log so that their curved surfaces are touching. Both logs are smooth. A horizontal force P at a height r above the floor pushes the smaller log in the direction perpendicular and towards the wall.

Prove that by increasing P to $\frac{4}{3}$W, the reaction between the larger log and the floor can be made zero. (H.C.)

10. A uniform heavy rod of length $4a$ and weight, W, can turn freely about one end A which is fixed; to this end is attached a string of length a which supports a shpere of radius a and weight W. Find the inclinations of the rod and the string to the vertical, the tension in the string, and the pressure between the rod and the sphere.

[*Ans.* 15°, 15°, $W(\sqrt{3}+1)/\sqrt{6}$; $W(\sqrt{3}-1)/\sqrt{6}$]

11. Show that the power necessary to move a cylinder, of radius r and weight W, up a smooth plane inclined at an angle α to the horizon by means of a light crowbar of length l inclined at an angle β to the horizon is $Wr \sin \alpha/l[1+\cos(\alpha+\beta)]$.

(*P. U.* 1922)

12. The ends of two uniform rods of weight W per foot and of length 6 and 8 ft. respectively are freely hinged, and the other ends rest on a smooth horizontal table. The middle points of the string are connected by a string 5 ft. long. Find

the tension of the string when the rods are supposed to stand in a vertical plane. [*Ans.* $6\frac{2}{3}$W. (*P. U.* 1942]

13. A pair of compasses rests across a smooth horizontal cylinder of radius r. Show that the frictional couple at the joint to prevent slipping must be W(r cot α cosec $\alpha - a$ sin α), where 2α is the angle between the legs and W the weight of each leg and a the distance of its C. G. from the joint.

7·4. Problems involving friction.

Example. (*i*). *A uniform ladder rests against a vertical wall with its lower end on a horizontal floor. Find the greatest angle the ladder can make with the vertical if the co-efficient of friction at the lower and upper ends are respectively* μ *and* μ'.

The ladder is supposed to rest in a vertical plane perpendicular to the wall.

Let AB be the ladder, $2a$ its length, W its weight acting at its middle point at G and θ its greatest inclination with the vertical. In this position the ladder is on the point of slipping such that the end A tends to move away from O.

If R and S be the normal pressures at A and B, the frictions μR and μ'S will act as shown.

Resolving forces horizontally and vertically, we have

$$S=\mu R \ldots\ldots(1), \quad R+\mu' S=W \ldots\ldots\ldots(2)$$

Taking moments about A, we get

$$S.\,2a\cos\theta+\mu' S.2a\sin\theta$$

$$=Wa\sin\theta \quad \ldots\ldots(3)$$

From (1) and (2), we have

$$R(1+\mu\mu')=W \quad \text{and} \quad S(1+\mu\mu')=\mu W.$$

$$\text{From (3)} \quad \tan\theta=\frac{2S}{W-2\mu' S}=\frac{2\mu W}{W(1+\mu\mu')-2\mu\mu' W}$$

$$=\frac{2\mu}{1-\mu\mu'} \quad \ldots\ldots(4)$$

Cor. (*a*) If the wall and the floor are equally rough, $\mu=\mu'$

and $$\tan\theta=\frac{2\mu}{1-\mu^2}=\frac{2\tan\lambda}{1-\tan^2\lambda}=\tan 2\lambda.$$

i.e., θ = twice the angle of friction.

(*b*) If the wall is smooth, $\mu'=0$

$\therefore$ $\tan\theta=2\mu=2\tan\lambda$.

(*c*) In case the wall and the floor are both smooth, $\mu=\mu'=0$, $\therefore\ \theta=0$ *i.e.*, the ladder cannot rest in an inclined position, which is otherwise obvious.

Example. (*ii*) *If the ladder rests in the limiting position as given by equation* (*4*), *how high can a man ascend without its slipping* ?

Let the ladder be on the point of slipping when the man has ascended a distance x. Let w be the weight of the man, the equations now become

$S=\mu R$,,.....(5) $\quad R+\mu' S=W+w$.....(6)

and $S.2a\cos\theta+\mu' S.2a\sin\theta=Wa\sin\theta+wx\sin\theta$, ...(7)

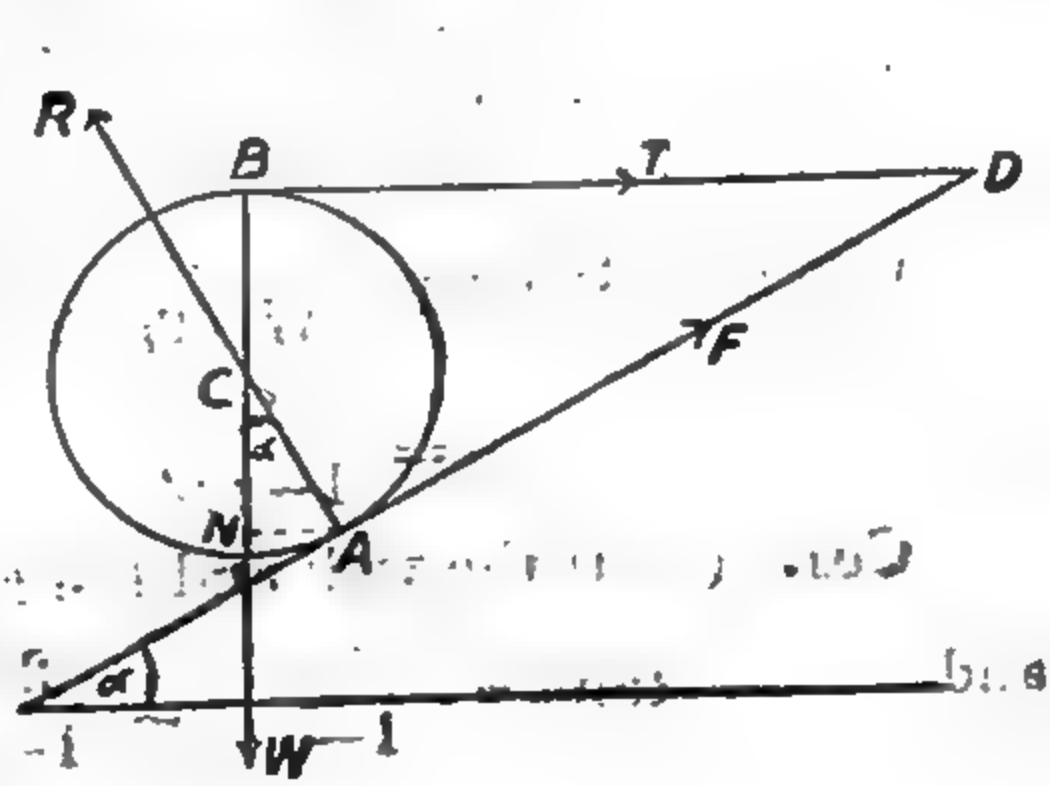
From (5) and (6) $S(1+\mu\mu')=\mu(W+w)$.

Substituting this value of S, and value of θ as given in equation (4), in equation (7), we obtain, after simplification, $x=a$.

So that the man, whatever his weight may be, eannot ascend more than half the length of the ladder.

Example. (*iii*) *A full reel of cotton of radius a and weight W is kept at rest with the axis horizontal on a plane of inclination* α, *by holding the thread which comes off horizontally and tangentially from the middle of the reel. Find the tension of the thread and friction between the real and the plane. Find also the least co-efficient of friction for which the reel will rest in this way.*

The real is full, so that it can be treated as a cylinder, the circle ABC is its vertical section through the middle, A being the point of contact with the plane, B the highest point and C the centre.



The forces acting on the reel are :—

the tension T acting hori-

zontally, meeting the plane in D, the weight W acting vertically at C, the resultant normal reaction R of the plane and the resultant frictional force F along the plane.

Taking moments about C, we have

$$T.CB=F.CA \quad i.e., \quad T=F \qquad (1)$$

Taking moments about D, we get

$$R.AD=W.BD \quad i.e., \quad R=W \qquad (2)$$

Taking moments about A, we have

$$W.AN=T.BN=T(BC+CN)$$

$$i\,e., \; Wa \sin \alpha = T(a+a\cos\alpha) \text{ or } T=\frac{\sin\alpha}{1+\cos\alpha}W=W\tan\frac{\alpha}{2}$$

$$\therefore \; F=T=W\tan\frac{\alpha}{2} \text{ and from (2) } F/R=\tan\frac{\alpha}{2}.$$

Now for equilibrium, $F<\mu R$ *i.e.*, $\mu>F/R$ *i.e.*, $>\tan\frac{\alpha}{2}$

Hence the least value of $\mu=\tan\frac{\alpha}{2}$.

Note. The solution has been simplified by taking moments about three non-collinear points.

EXAMPLES XXVII

1. A uniform ladder of length 15 ft. and mass 50 lb. is placed with its upper end against a smooth vertical wall and its lower end on a rough horizontal plane at a distance of 3 ft. from the wall; a man of mass 10 stone stands at the top. Find the least co-efficient of friction in order that the ladder shall not slip [*Ans.* ·177]

2. A man weighing 140 lb. climbs up a uniform ladder, 20 ft. long and 70 lb. in weight, which rests against a rough vertical wall at an angle of 45°. If the co-efficient of friction at each end of the ladder is $\frac{1}{2}$, how far will the man be able to climb up the ladder before it begins to slip? [*Ans.* 13 ft.

3. A ladder which stands on horizontal ground, leaning against a vertical wall, is so loaded that its centre of gravity is at distances *a* and *b* from its lower and upper ends respectively. Show that if the ladder is in limiting equilibrium, its

inclination θ to the horizontal is given by $\tan\theta = \dfrac{a-b\ \mu\mu_1}{(a+b)\mu}$

where μ, μ_1 are the co-efficients of friction between the ladder and the ground and between the ladder and the wall. (*D.U.* 1940)

4. A uniform ladder of weight W is placed with one end resting against a smooth vertical wall and the other end on rough horizontal floor, the ladder making an angle α with the wall. Show that if the co-efficient of friction μ is less than $\frac{1}{2}\tan\alpha$, the ladder cannot remain in equilibrium alone, but that equilibrium may be maintained by placing a rough block on the floor at the foot of the ladder, provided that its weight exceeds $W\left(\dfrac{1}{2\mu}\tan\alpha - 1\right)$ the co-efficient of friction between the block and the floor being also μ. (*H. C.*)

5. A uniform rod of weight W rests with its one end against a rough inclined plane AB, of inclination α, and the other end against a smooth vertical wall BD, B being higher than A. If θ be the inclination of the rod to the vertical in the limiting position of equilibrium, show that

$$\tan\theta = 2\tan(\lambda - \alpha).$$

where λ is the angle of friction for the plane. [Compare Ex. 1 Art. 7·2].

6. A uniform rod is in limiting equilibrium, one end resting on a rough horizontal plane and the other on an equally rough plane inclined at an angle α to the horizon. If λ be the angle of friction and the rod be in a vertical plane show that the inclination θ of the rod to the horizon is given by (*D. U.* 1939)

$$\tan\theta = \frac{\sin(\alpha - 2\lambda)}{2\sin\lambda\ \sin(\alpha - \lambda)}$$

7. A uniform ladder, of length l and weight W, rests with its foot on rough ground, and its upper end against a smooth vertical wall, the inclination of the ladder with the vertical being α. A force P is applied horizontally to the ladder at a point distant c from the foot, so as to draw the foot to the

wall. If μ the co-efficient of friction at the foot, prove that P must exceed

$$\frac{Wl}{l-c}(\mu+\tfrac{1}{2}\tan\alpha). \qquad (P.\ U.\ 1942)$$

8. A uniform rod of length l rests in a vertical plane against (and over) a smooth horizontal bar at a height h, the lower end of the rod being on level ground. Show that if the rod is on the point of slipping when its inclination to the horizontal is θ then the co-efficient of friction between the rod and the ground is $\dfrac{l\sin 2\theta\sin\theta}{4h-l\sin 2\theta\cos\theta}$.

9. A uniform ladder of length $4a$ rests at an angle α to the horizontal against a smooth horizontal rail at a height h from the ground. If λ be the angle of friction between the ground and the ladder, show that a man of twice the weight of the ladder may ascend a distance

$$3h\sin\lambda\ \mathrm{cosec}\ (\alpha+\lambda)\ \mathrm{cosec}\ 2\alpha-a. \qquad (D.\ U.\ 1942)$$

10. One end A of a heavy uniform rod AB can slide along a rough horizontal rod AC to which it is attached by a ring; B and C are joined by a string. If ABC be a right angle when the rod is on the point of sliding, μ the co-efficient of friction and α the angle between AB and the vertical, show that

$$\mu=\frac{\tan\alpha}{2+\tan^2\alpha} \qquad (D.\ U.\ 1940)$$

11. A heavy circular disc whose plane is vertical is kept at rest on a rough inclined plane by a string parallel to the plane and touching the circle, show that the disc will slip on the plane if the co-efficient of friction be less than $\frac{1}{2}\tan\alpha$, where α is the slope of the plane. $(P.\ U.\ 1934)$

12. A sphere of weight W is placed on a rough plane, inclined to the horizontal at an angle θ which is less than the angle of friction, show that a weight $\dfrac{W\sin\theta}{\cos\theta-\sin\theta}$, fastened to the sphere at the upper end of the diameter which is parallel to the plane, will just prevent the sphere from rolling down the plane. $(P.\ U.\ 1939)$

13. A uniform rod slides with its ends on two fixed equally rough rods, one being vertical and the other inclined at an angle α to the horizon. Show that the inclination θ to the horizon of the moveable rod, when it is on the point of sliding, is given by

$$\tan\theta = \frac{1 \mp 2\mu \tan\alpha - \mu^2}{2(\tan\alpha \pm \mu)}.$$

14. A heavy rod of length $2a$ lies over a rough peg with one extremity leaning against an equally rough vertical wall. If a be the distance of the peg from the wall and μ the co-efficient of friction, show that when the point of the rod in contact with the wall is above the peg, then

$$\sin^3\theta = \cos^2\lambda$$

where θ is the inclination of the rod to the wall and λ the angle of friction.

15. A uniform rod AB of weight W, rests with the end B against a rough vertical wall and is held by a string OA, of the same length as the rod, fastened to a point O vertically about B. If the rod is on the point of sliding down, find its inclination to the vertical and the tension of the string.

[*Ans.* $\cot^{-1}\frac{1}{3}\mu$; $\frac{1}{4}W\sqrt{1+9/\mu^2}$.

16. Two small fixed pegs are in a line inclined at an angle θ to the horizon. A rough thin rod passes under the lower and rests on the higher, this latter peg being lower than the C.G. of the rod. The distances of the C. G. from the two pegs are a and b respectively and the co-efficient of friction is μ. Show that if the rod is on the point of motion,

$$\mu(a+b) = (a-b)\tan\theta.$$

17. A uniform rod rests on two rough pegs A and B ; AB is inclined at an angle α to the horizon and the co-efficients of friction are μ_1 and μ_2 respectively. If the rod is in limiting equilibrium, prove that its middle point divides AB in the ratio $(\mu_1 - \tan\alpha) : (\tan\alpha - \mu_2)$, it being given that

$$\mu_1 > \tan\alpha > \mu_2.$$

18. A uniform rod of length $2a$ and weight W is placed against a smooth fixed cylinder with one end A resting on a rough horizontal plane, which passes through the centre of the cylinder. The rod makes an angle θ with the ground and d is the distance from the centre of the cylinder of the lower end of the rod. Show that if the rod is on the point of slipping

$$\frac{P}{W}=\frac{\sin\phi}{\sin(\theta+\phi)}=\frac{a}{d}.$$

where ϕ is the angle of friction, and P, the pressure between the rod and the cylinder.

19. In the last example, if the cylinder is rough instead of smooth and ϕ' be the angle of friction for it, prove that

$$a\sin\theta\sin(\theta-\phi'+\phi)=r\sin\phi\cos\phi'$$

and show that $\tan\theta>r/2a$, r being the radius of the cylinder.

20. A uniform plank of length $2l$ rests with one end A on a rough horizontal plane and across a fixed smooth cylinder, of a radius r, lying on the same plane. The plank makes an angle 2α with the plane and λ is the angle of friction. Show that the equilibrium is possible if

$$r\sin\lambda>l\tan\alpha\cos2\alpha\sin(2\alpha+\lambda) \qquad (All.\ U.)$$

21. AB and AC are two equally rough wires of wich AB is horizontal and AC vertical Two rings of equal mass connected by a light inextensible string are free to move on AB and AC. Prove that in the position of limiting equilibrium the string makes an angle with the vertical equal to twice the angle of friction and show that the tension of the string in this position is equal to the weight of each ring.

7.41. A rod resting inside a rough sphere. *A heavy rod, whose C. G. divides it into two parts a and b, rests inside a rough spherical bowl (in a vertical plane through the centre). Show that the greatest possible inclination θ, of the rod to the horizontal is given by*

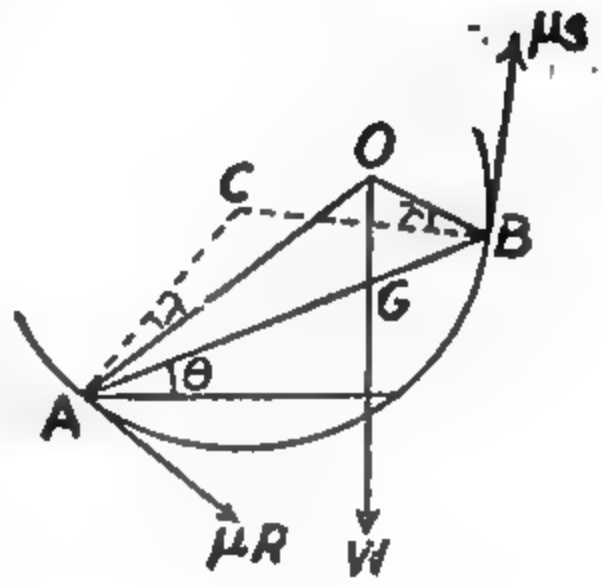
$$(a+b)\tan\theta=b\tan(\alpha+\lambda)-a\tan(\alpha-\lambda)$$

where λ is the angle of friction.

Let AB be the rod, G its C. G., AG$=a$, GB$=b$ and θ the

...GIRLS COLLE...

greatest possible angle which the rod makes with the horizontal, so that in this position it is on the point of sliding down.



At the point A let the resultant of the normal pressure R and the tangential ftiction μR act along AO, making an angle λ with the normal AC. Similarly at B the resultant reaction acts along BO, making an angle λ with the normal BC, Cbeing the centre of the bowl.

Since these reactions meet at O, the vertical line through G, along which the weight acts, must also pass through O.

Now $\angle ACB = 2\alpha$ $\therefore$ $\angle CAG = \angle CBG = 90 - \alpha$

$\therefore$ $\angle OAG = \angle CAG - \angle CAO = 90 - \alpha - \lambda$,

and $\angle OBG = \angle CBG + \angle CBO = 90 - \alpha + \lambda$,

also $\angle OGB = 90 - \theta$, AG $= a$ and GB $= b$.

Using the trigonometrical formula (2) Art. 6·41, we have

$$(a+b)\cot \angle OGB = b \cot \angle OAG - a \cot \angle OBG$$

$$\therefore \quad (a+b)\tan\theta = b\tan(\alpha+\lambda) - a\tan(\alpha-\lambda)$$

In particular, when the rod is uniform *i.e.*, $a = b$, we get

$$2\tan\theta = \tan(\alpha+\lambda) - \tan(\alpha-\lambda)$$

which gives

$$\tan\theta = \frac{\sin 2\lambda}{\cos 2\alpha + \cos 2\lambda}.$$

EXAMPLES.

1. A *uniform* rod of length $2a \sin\alpha$ rests within a rough vertical circle of radius a, show that the greatest possible inclination of the rod to the horizon is

$$\tan^{-1}\frac{\mu}{\cos^2\alpha - \mu^2\sin^2\alpha}$$

where μ is the co-efficient of friction.

2. A heavy uniform rod rests within a rough circular ring fixed in a vertical plane; the rod subtends an angle of 120° at

the centre of the ring and in the limiting position of equilibrium is inclined to the horizon at an angle α. If $\sqrt{3}\mu=\tan\phi$, μ being the co-efficient of friction, show that

$$\tan\alpha : \tan 2\phi = 2 : \sqrt{3}.$$

3. A uniform rod is placed inside a rough cylindrical drum which is fixed with its axis horizontal. If the rod rests in a vertical plane perpendicular to the axis of the drum, show that its least possible inclination to the vertical is

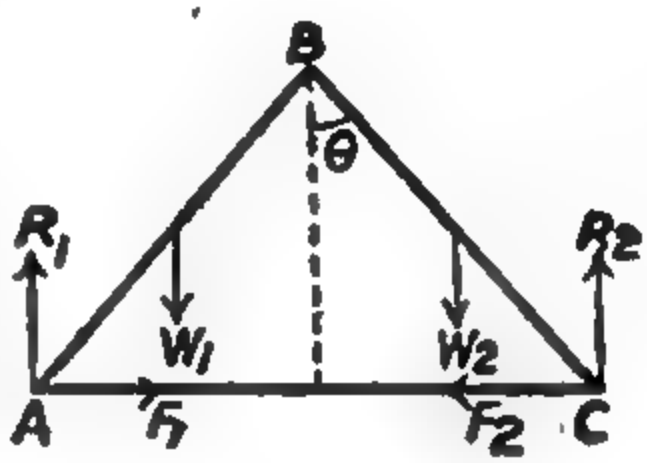
$$\tan^{-1}(\cot 2\lambda + \cos\beta \operatorname{cosec} 2\lambda)$$

where λ is the angle of friction, and β the angle the rod subtends at the nearest point of the axis of the drum.

*** 7.42. Two Jointed rods standing on a rough table.** *Two uniform rods AB, BC each of length 2a and of weights W_1 and W_2 respectively ($W_2 > W_1$) are freely jointed together at B. The system stands in a vertical plane, with A and C on a rough horizontal table, each rod making an angle θ with the vertical The co-efficient of friction at both A and C is μ. Show that equilibrium is possible only if*

$$(W_1+W_2)\tan\theta \leqslant (3W_1+W_2)\mu.$$

Let R_1, R_2 be the normal reactions and F_1, F_2 the forces of friction at A and C.



Since the reactions at B on the two rods are equal and opposite, they cancel each other when the system is taken as a whole.

$\therefore$ taking moments abont A, for both the rods, we have

$$R_2.4a\sin\theta = W_2 3a\sin\theta + W_1 a\sin\theta, \text{ i.e., } R_2 = \frac{W_1+3W_2}{4}.$$

Similarly by taking moments about C, $R_1 = \dfrac{3W_1+W_2}{4}$

Now taking moments about B, for the rod BC, we get

$$F_2.2a\cos\theta + W_2 a\sin\theta = R_2\, 2a\sin\theta.$$

which gives $F_2 = (R_2 - \frac{1}{2}W_2)\tan\theta = \dfrac{W_1+W_2}{4}\tan\theta.$

Similarly $F_1 = \frac{W_1+W_2}{4}\tan\theta$.

$\therefore \frac{F_1}{R_1} = \frac{W_1+W_2}{3W_1+W_2}\tan\theta$ and $\frac{F_2}{R_2} = \frac{W_1+W_2}{3W_2+W_1}\tan\theta$.

Since $W_2 > W_1$, $\therefore \frac{F_1}{R_1} > \frac{F_2}{R_2}$, so that $\frac{F_1}{R_1}$ will be first to attain the maximum limit μ, *i.e.*, the end A of the rod AB will be the first to slip.

Hence the equilibrium of the system will be possible so long as $\frac{F_1}{R_1} \leqslant \mu$.

i.e., $(W_1+W_2)\tan\theta \leqslant (3W_1+W_2)\mu$.

EXAMPLES

1. In the above example prove that the reaction at the hinge B is inclined at

$$\tan^{-1}\frac{W_2-W_1}{W_2+W_1}\cot\theta$$ with the horizontal.

2. Two uniform rods AB, BC of equal lengths and weights W and 2W respectively, are smoothly hinged at B and rest in a vertical plane with the ends A and C on a rough horizontal plane. If the angle ABC is a right angle, find the horizontal and vertical components of the reactions at A and C. Find also the least value of the co-efficient of friction between the rods and the table in order that the rods may be able to rest in this position.

[*Ans.* $\frac{3}{4}$ W at A and C ; $\frac{5}{4}$ W at A, $\frac{7}{4}$ W at C ; $\frac{3}{5}$].

3. Two uniform ladders OA, OB of the same length l and of the same weight W are smoothly jointed together at O, and stand with A and B in contact with a rough horizontal plane ; μ is the co efficient of friction between the plane and either end A or B. If a man of weight W can stand anywhere on the ladder when A and B are at a distance $2a$ apart, prove that μ must not be less than $\frac{2a}{3\sqrt{l^2-a^2}}$.

4. Two equal uniform ladders are freely jointed at one end and stand with the other ends on a rough horizontal plane. A man whose weight is equal to that of one of the ladders ascends one of them. Prove that the other will slip first.

If it begins to slip when the man is half way up, prove that the co-efficient of friction is $\frac{3}{5}\tan\alpha$ where 2α is the angle between the ladders.

5. Two uniform heavy rods of length a are jointed together by a smooth hinge and placed symmetrically over two smooth pegs at a distance b apart in a horizontal line If in the position of equilibrium, each rod is inclined at an angle θ to the horizontal, prove that $b=a\cos^3\theta$.

If the pegs are rough and the rods are still placed symmetrically over the pegs, show that equilibrium is possible if and only if

$$a\cos^2\theta(\cos\theta-\mu\sin\theta)<b<a\cos^2\theta(\cos\theta+\mu\sin\theta).$$

6. Two similar uniform straight rods AB, and BC, each of length $2a$ and weight W, smoothly jointed at B, are in a vertical plane and rest in equilibrium on two rough pegs P and Q where P is the middle point of AB and Q is at a distance $\frac{2}{3}a$ from B along BC. The rods are inclined to the horizon at an angle α where $\tan\alpha=\frac{1}{2}$, and B is above the pegs. Calculate the normal components of the forces on the rods P and Q and show that the frictional component acts up one rod and down the other.

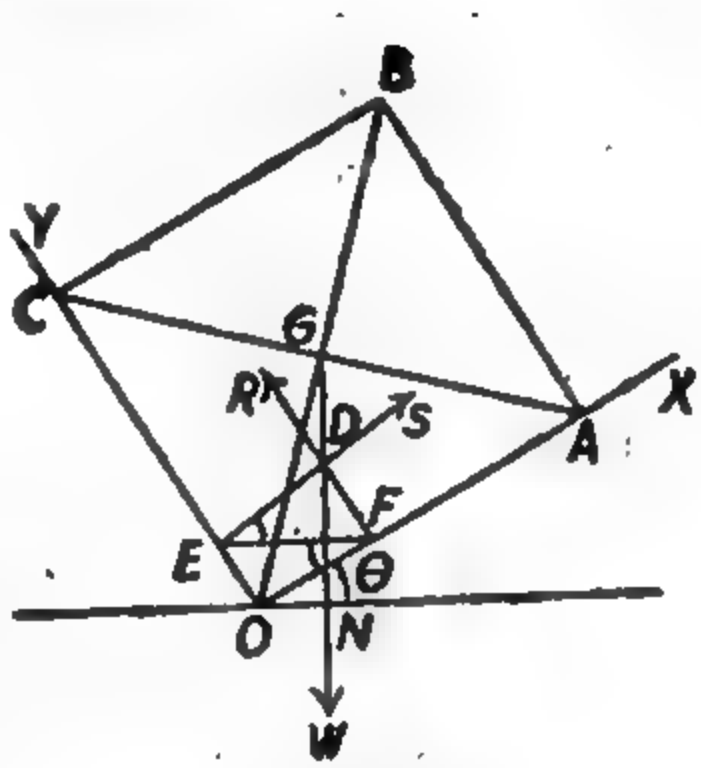
If both rods are on the point of slipping, find the co-efficient of friction at each of the two pegs.

[*Ans.* $2W/\sqrt{5}$, $3W/\sqrt{5}$; $\frac{1}{8}$, $\frac{1}{12}$].

Miscellaneous Examples III. A

1. *A square of side $2a$ is placed with its plane vertical between two smooth pegs which are in the same horizontal line and at a distance c, show that it will be in equilibrium when the inclination of one of its edges to the horizon is either 45° or* $\frac{1}{2}\sin^{-1}\left(\frac{a^2-c^2}{c^2}\right)$. *(All. U. ; P.U.*

Let OABC be a square, E and F the pegs, and θ the angle OA ($=2a$) makes with the horizontal. The reaction S and R at E and F and the weight meet at D.



Since $EF=c$, $OF=c\cos\theta$,
$OE=c\sin\theta$

We solve this problem in the following four ways :—

First Method. *Analytical.* Let OA, OC be taken as the co-ordinate axes, then the co-ordinates of G are (a, a) and those of
D$(c\cos\theta, c\sin\theta)$.

$\therefore$ the slope of the line $DG=\dfrac{a-c\sin\theta}{a-c\cos\theta}$ and since EF makes an angle $180°-\theta$ with the positive direction of OX, its slope is $\tan(180°-\theta)=-\tan\theta$.

Now since DG and EF are perpendicular, we have

$$\frac{a-c\sin\theta}{a-c\cos\theta}(-\tan\theta)=-1$$

i.e., $$(a-c\sin\theta)\sin\theta=(a-c\cos\theta)\cos\theta$$

$\therefore$ $$a(\cos\theta-\sin\theta)=c(\cos^2\theta-\sin^2\theta) \quad \ldots\ldots(1)$$

This gives either $\cos\theta-\sin\theta=0$ *i.e.*, $\theta=\dfrac{\pi}{4}$

or $$c(\cos\theta+\sin\theta)=a \quad \ldots\ldots(2)$$

Squaring (2), we have

$$c^2(1+\sin 2\theta)=a^2 \text{ } i.e., \text{ } c^2\sin 2\theta=a^2-c^2$$

$$\therefore \quad \theta=\tfrac{1}{2}\sin^{-1}\frac{a^2-c^2}{c^2}.$$

Second Method. *Statical* (*Art. 7·1*). Resolving forces prep. to OA, we have

$$R=W\cos\theta$$

Taking moments about E, we have

$$R.DE = W(EO \sin\theta + ON)$$

N being the point where GW intersects the horizontal line through O

i.e., $R.OF - W.EO \sin\theta = W.OG \cos(45^\circ + \theta)$

$\therefore$ $W \cos\theta . c\cos\theta - W.c \sin\theta . \sin\theta = W\sqrt{2}\, a \cos(45^\circ + \theta)$

i.e., $c(\cos^2\theta - \sin^2\theta) = a(\cos\theta - \sin\theta)$, the same as (1)

Third Method. *Geometrical.* By equating the length of ON obtained in two *different ways*, *viz.*, (*i*) horizontal projection of OG and (*ii*) the sum of the horizontal projections cf OF and FD. Thus we have

$$OG \cos(45^\circ + \theta) = OF \cos\theta - FD \sin\theta = c\cos^2\theta - c\sin^2\theta$$

$\therefore$ $a(\cos\theta - \sin\theta) = c(\cos^2\theta - \sin^2\theta)$, the same as (1).

Fourth Method. *By equating to zero, the differential of the height of the C. G. from some fixed horizontal leval.* (This is to be explained in Chap. IX.)

Here EF is the fixed horizontal level, z, height of G above EF = height of G above O — height of EF above O.

i. e. $z = OG \sin(45^\circ + \theta) - OF \sin\theta$

$= \sqrt{2a} \sin(45^\circ + \theta) - c\cos\theta \sin\theta$

$dz = \sqrt{2a} \cos(45^\circ + \theta) - (c\cos^2\theta - c\sin^2\theta)$

This equated to zero gives the same result as (1).

Note. Equation (2) (p. 154) can also be written as

$$c\sqrt{2} \sin(\theta + 45^\circ) = a$$

which gives

$$\theta = \sin^{-1}\frac{a}{c\sqrt{2}} - \frac{\pi}{4},$$

a from in which the answer is sometimes required.

2. A triangular lamina ABC has a right angle at C and rests in a vertical plane with its sides in contact with two smooth pegs D, E in the same horizontal plane, the vertex C being downwards. Prove that if, in equilibrium, AC makes an angle θ with the horizon,

$$AC \cos\theta - BC \sin\theta = 3\, DE \cos 2\theta.$$

3. Two smooth planes are each inclined at 45° to the upward vertical their line of intersection being horizontal. A cube of weight W and edge a is placed with one edge on each plane and its lowest face inclined at an angle θ to the horizontal. Show that the magnitude of the couple required to keep the cube in position is $W a \sin \theta$.

4. AB, AC are two rods, each of length $2a$ and weight 4W, freely jointed at A, with B and C joined by two light strings, each of length $2a$, to a particle of weight W. The system rests in equilibrium with the rods symetrically placed over two smooth pegs, in the same horizontal line and distant $2c$ apart. Show that the inclination of either rod to the vertical is given by

$$\sin \theta = (3c/4a)^{\frac{1}{3}}.$$

5. AB, BC, CD are three equal rods freely joined at B and C. The rods AB, CD rest on two pegs in the same horizontal line so that BC is horizontal. If α be the inclination of AB, ann β the inclination of the reaction at B to the horizon, prove that $3 \tan \alpha \tan \beta = 1$.

6. A uniform rod AB of length $2a$ rests with the end A in contact with a smooth plane inclined at an angle α to the horizon. A point C of the rod rests against a smooth horizontal rail perpendicular to a line of greatest slope of the plane. Prove that, if the inclination of the rod to the vertical is θ, then

$$a \sin \theta \cos^2 (\theta - \alpha) = b \sin \alpha.$$

where b is the perpendicular distance of C from the plane.

7. A uniform rod AB is smoothly hinged to a point A in a smooth vertical wall; at B is smoothly hinged another rod BC of half the length of AB and half its weight. The end C is in contact with the wall, (the rods being in a vertical plane perp. to the wall). Prove that in the position of equilibrium the upper rod makes with the wall an angle $\sin^{-1} \frac{1}{\sqrt{5}}$.

Calculate the pressure at C and the reactions at the hinges A and B.

[*Ans* $\frac{1}{2}$ W ; W$\sqrt{10}/2$; W$\sqrt{2}/2$.

8. A rod AB of length l and weight W is hinged to a fixed point at A and a string of length $l/\sqrt{3}$ connects its end B to a fixed point C which is at a height $\sqrt{2}\ l/3$ vertically above it. Find what weight must be suspended from the middle point of the string BC in order that the rod may be in equilibrium with B and C at the same level. [*Ans.* W/2.

9. Two smooth spheres of weights W, W′ rest in contact in the angle between two smooth planes of inclinations α, α', respectively. Prove that in the position of equilibrium, the angle θ made by the line joining the centres with the vertical is given by

$$(W+W')\cot\theta = W'\cot\alpha - W\cot\alpha'$$

Given $W'=20$ lb., $W=10$ lb., $\alpha=30°$, $\alpha'=45°$, find θ.

10. Two heavy uniform cylinders of radius a and weight W are placed in contact on the same level inside a smooth fixed cylinderical shell of radius $3a$; a third equal cylinder is placed on the other two, the axes of all the cylinder being horizontal and parallel Show that the pressure between the two lower cylinders is $W/\sqrt{3}$.

11. Two equal heavy cylinders of radius r placed in contact on the same level inside a smooth fixed cylinder of radius $R\ (>2r)$; a third equal cylinder of radius r is placed on the other two, the axes of all the cylinders being horizontal and parallel. Show that the two lower cylinders will not spearate if $R<r(1+2\sqrt{7})$.

12. Two wheels of radius R are keyed to a co-axial shaft of radius r, the whole weighing W lb. The wheels rest with the shaft horizontal on a rough inclined plane, which makes an angle α with the horizontal. The wheels are prevented from rolling down the plane by a light cord. One end of the cord is fixed to a point P on the plane and the part of the cord is wrapped round the shaft, the part of the chord between P and the shaft is perp. to the axis of the shaft and makes an acute angle β with the inclined plane.

Consider the two possible cases, and show that the pull in the cord is either

$$\frac{WR\sin\alpha}{R\cos\beta+r}\text{ lb.}\quad\text{or}\quad\frac{WR\sin\alpha}{R\cos\beta-r}\text{ lb.}\quad(\textit{Roorkee}),$$

13. Forces P. Q, R acting along the sides BC, CA, AB of a triangle ABC are in equilibrium with forces P′, Q′, R′ acting along AG, BG, CG, G being the C. G. of the triangle. Prove that

$$\frac{PP'}{AG.BC}+\frac{QQ'}{BG.CA}+\frac{RR'}{CG.AB}=0.$$

14. Three uniform rods VA, VB, VC of total weight W are connected at V by a smooth weightless joint, and rests in the form of a tripod on a rough horizontal table with their other ends at A, B, C points on the table. The projection of V on the table is O, and the angles BOC, COA, AOB are respectively α, β, γ, also the rods make angles θ, ϕ, ψ respectively with the vertical. Prove that the frictions F_1, F_2, F_3, at A, B, C are given by

$$\frac{F_1}{\sin\alpha}=\frac{F_2}{\sin\beta}=\frac{F_3}{\sin\gamma}=\frac{1}{2}\frac{W}{\sin\alpha\cot\theta+\sin\beta\cot\phi+\sin\gamma\cot\psi}$$

(*D. U.* 1938)

Miscellaneous Examples III B

(μ stands for the co-efficient of friction, λ for the angle of friction).

1. A *uniform triangular plane ABC, of weight W, rests on a horizontal table, supported by three small pegs at the corners A, B ,C. What fraction of the weight is borne by each peg ?*

If the peg at A fits into a small smooth hole in the table, and if μ is the co-efficient of friction at B and C, show that the least force acting along the side BC, that will cause the plate to rotate about A is $\mu a(b+c)W/6\triangle$, where $\triangle$ denotes the area of the triangle.

By taking moments about the sides, it can be seen that the reaction at each vetex is equal to $\frac{1}{3}$W.

Suppose the plate is just on the point of rotation when a force P acts along BC.

Frictions of amount $\frac{1}{3}\mu$ W at B and C acting perpendicular to AB and AC oppose rotation.

Taking moments about A, we have

$$P.\frac{2\Delta}{BC}=\tfrac{1}{3}\mu W.AB+\tfrac{1}{3}\mu W.AC$$

which gives $P=\tfrac{1}{6}\mu a(b+c)W/\Delta$.

2. (*a*) A circular cylinder of radius *a* and weight W rests with its axis horizontal on a rough horizontal plane in contact with an equally rough vertical surface. Show that the least value of the couple to rotate the cylinder is $\mu(1+\mu)aW/(1+\mu^2)$.

(*b*) If the cylinder is placed on two equally rough planes, each inclined at an angle α to the horizontal, the least couple to turn it about its axis is $\frac{1}{2}Wa$ sin 2λ sec α.

3. A circular cylinder of radius *a* and weight W rests with its axis horizontal on a rough inclined plane of inclination α. Show that the cylinder cannot be held at rest by a couple alone if $\mu < \tan\alpha$ and find the moment of the couple that will hold it at rest if $\mu > \tan\alpha$. (*Ans.* Wa sin α

4. A uniform sphere rests on a rough plane of inclination α; the highest point of the sphere being attached by a horizontal string to a point in the plane. Find the inclination of the plane if the sphere is on the point of slipping, the angle of friction being λ. Find also the ratio of the tension of the string to the weight of the sphere [*Ans.* 2λ, tan λ]

5. A sphere rests on a rough horizontal plane and its highest point is joined to a peg fixed in the plane by a tight cord parallel to the plane. Show that, if the plane be gradually tilted about a line in it perpendicular to the direction of the cord, the sphere will not slip until the inclination becomes equal to $\tan^{-1}2\mu$, where μ is the co-efficient of friction.

6. A circular cylinder of radius *a* has its centre of gravity G at a distance $\frac{a}{2}$ from the middle point O of the axis of the cylinder and equidistant from the plane ends. The cylinder rests in equilibrium on a rough inclined plane with OG horizontal, the axis of the cylinder being horizontal. Find the inclination of the plane to the horizontal and the least permissible value of the co-efficient of friction. [*Ans.* $30°, \frac{1}{\sqrt{3}}$

7. A homogeneous hemisphere rests in limiting equilibrium with its curved surface on a rough inclined plane such that its plane face is vertical. Show that the co-efficient of friction is $3/\sqrt{55}$. (*P. U.* 1928)

8. A sphere, whose radius is a and whose C. G. is at a distance c from the centre O rests in limiting equilibrium on a rough plane inclined at an angle α to the horizon. Show that it may be turned through an angle $2\cos^{-1}\left(\frac{a \sin \alpha}{c}\right)$ and still be in equilibrium. (*Agra U.*)

9. A bow is formed of an arc AB of a circle, which subtends an angle 2α at the centre O of the circle, and its chord (both having the same mass per unit length). If the bow be hung with its arc over a rough peg, prove that the point of contact may be anywhere in the arc CD, which subtends an angle 2β at G, the C.G. of the whole bow, such that

$$\sin \beta = \frac{(\alpha + \sin \alpha) \sin \lambda}{(1 + \cos \alpha) \sin \alpha}$$

provided that λ makes this fraction less than unity. (*B. U.*)

[For C. G. of the bow see Ex. Art. 8·6].

10. If the segment of a sphere of radius a and height $a/2$ is supported with its curved surface on a needle point, show that the greatest angle at which the base can be inclined to the horizontal, so as to make equilibrium possible is $\sin^{-1}(\frac{40}{27} \sin\lambda)$ where λ, the angle of limiting friction is supposed to be less than $\sin^{-1}\frac{27}{40}$.

[For C. G. of the segment see Cor. Art. 8·71].

11. A uniform circular ring of weight W and radius a hangs in a vertical plane from two rough pegs in the same horizontal plane at a distance $2b$ apart, where b is less than a. Show that it is possible to fix a suitable weight w to the ring in such a way that one peg may be removed without disturbing the ring, if, and only if, the angle of friction at the remaining peg is not less than $\sin^{-1}(b/a)$.

Find the value of w, it is attached at the end of a horizontal diameter.

Ans. $\frac{Wb}{a-b}$

12. (*a*) A solid homogeneous hemisphere rests on a rough horizontal plane against a smooth vertical wall. Show that, if the co-efficient of friction be greater than $\frac{3}{8}$, the hemisphere can rest in any position and, if it be less, the least angle that the base of the hemisphere can make with the vertical is

$$\cos^{-1}\frac{8\mu}{3}.$$

(*P. U. Hons.*)

12. (*b*) A solid hemisphere rests with its base in an inclined position at an angle θ to the horizontal, its curved surface resting on a horizontal plane and against a vertical plane. If the hemisphere is on the point of slipping, show that

$$c \sin \theta/a = \mu(1+\mu')(1+\mu\mu')$$

where a is the radius, c the distance of the C. G. on the axis of symmetry from the centre and μ, μ' are the co-efficients of friction for horizontal and vertical planes.

If $\mu=\mu'$ and $c=3a/8$, show that there is no position of equilibrium for the hemisphere if $5\mu > \sqrt{31}-4$.

13. A heavy ball has a fine vertical hole drilled through it, along a chord subtending an angle 2α, at the centre, and it can slide up and down a vertical wire which fits loosely in the hole. If λ be the angle of friction, prove that the ball can remain in equilibrium, without any other support, if α is not greater than λ.

(*All U.*)

14. (*a*) A drawer of length l is to be pulled out by a single force P parallel to the drawer. Show that if the force is applied at a distance greater than $l/2\mu$ from the middle of the end, the drawer will not move however great P may be.

14. (*b*) A light shallow rectangular drawer is of width a, while its length from front to back is l. It can be pulled out by two knobs attached to the front at equal distances f from the sides. If a tension parallel to the sides is applied to only one of the knobs, prove that the drawer will jam, if the co-efficient of friction between the sides and casing exceeds $l/(a-2f)$.

(*I. S. E.*)

15. A uniform girder AB of length l is being lifted from the ground by means of a chain fastened to the end A and passing over a fixed pulley at the height h $(>l)$ vertically above the

initial position of A on the ground. If θ is the inclination of the girder to the horizontal when the end B begins to slip, prove that the co-efficient of friction μ between the girder and the ground is given by equation

$$\mu[h+l\sin\theta-2l\tan\theta]=l(1-\cos\theta).$$

16. A uniform bar AB rests with its ends on two equally rough planes of equal inclination α which face each other, in a plane perp. to their line of intersection. If $\lambda<\alpha$, prove that the limiting inclination θ of the bar to the horizontal is given by

$$\tan\theta=\frac{\sin 2\lambda}{2\sin(\alpha+\lambda)\sin(\alpha-\lambda)}.$$

17. A uniform rod of length $2l$, one end of which is on a rough horizontal plane, rests against a smooth horizontal rail at the point which is distant $\frac{1}{2}$ from the other end. The rod is at right angle to the rail and is inclined at an angle of 45° to the horizontal. If μ is the co-efficient of friction besween the rod and the horizontal plane, prove that $\mu\geqslant\frac{1}{2}$.

If a gradually increasing horizontal force directed away from the rail is applied at the point of the rod distant $\frac{1}{2}l$ from the lower end in the vertical plane containing the rod, prove that the rod will ultimately rotate about its lower end or will slide according as μ is greater than or less than 2.

18. State the laws of limiting friction. A uniform rod of length $2a$ rests with one end A on a rough plane of inclination α to the horizontal, and lies in a vertical plane through a line of greatest slope of the plane. It is supported by a smooth peg at distance c from the end in contact with the plane. Show that if the end of the rod is about to slip up the plane and λ is the angle of friction, the inclination θ of the rod to the horizontal is given by

$$c\sin(\alpha+\lambda)=a\cos\theta\sin(\theta+\alpha+\lambda). \qquad (D.U.\ 1941)$$

19. A small ring of weight w, fixed to one end A of a uniform rod AB of weight W, can slide along a rough rod CAX which is fixed horizontal. A string CB of length equal to AB connects B to a fixed point C in CAX. Prove that AB can be in equilibrium inclined at an angle θ to the vertical if

$$W\tan\theta<\mu(3W+4w)$$

where μ is the co-efficient of friction between the ring and the rod CAX.

20. A trolley has four wheels and the axles $2a$ apart. The C. G. is half way between the axles and h above the ground when the trolley stands on level. It is found that the trolley just slips down a slope θ with the upper wheels locked and down a slope ϕ when the lower wheels alone are locked. Show that

$$\tan\ \theta=\frac{\mu a}{2a+\mu h} \text{ and } \tan\ \phi=\frac{\mu a}{2a-\mu h}. \quad (Roorkee)$$

21. A uniform bar ABCD of length $5a$ rests at B and C across two rough parallel horizontal rails which are at the same level and distant $2a$ apart, so that $AB=BC=2a$, $CD=a$. The co-efficients of friction at B and C are μ_1 and μ_2 respectively. If a slowly increasing force is applied at A parallel to the rails, show that equilibrium is broken by slipping at B if $2\mu_1>3\mu_2$.

22. Two equally rough pegs A and B are at a distance $2a$ apart in a straight line inclined at an angle θ to the vertical. A rod passes over the peg A and under the peg B and is just kept from sliding down by the friction at the pegs. Prove that the C. G. of the rod must be at a distance $a\left(\frac{\cot\theta}{\mu}-1\right)$ from the upper peg A, where μ is the co-efficient of friction between the rod and the pegs.

23. A thin uniform rod passes over one peg and under another, the coefficient of friction between each peg and the rod being μ. The distance between the pegs is a and the line joining them makes an angle β with the horizontal. Show that equilibrium is not possible unless the length of the rod is greater than $\frac{a}{\mu}(\mu+\tan\beta)$. (*P.U.* 1926)

24. A straight uniform rod weighing 10 lb. of negligible cross section rests along a line of greatest slope on a plane inclined to the horizontal at an angle of 60°. The angle of friction between the rod and the plane is 30°. The rod is prevented from slipping by a horizontal force of 10 lb. wt. applied at the upper end and in the vertical plane of the rod

Find the magnitude of the resultant reaction of the plane and the angle which this reaction makes with the rod.

[*Ans.* $10\sqrt{2}$ lb. 75°.]

25. Two rough particles connected by a light string rest on an inclined plane. If their weights and corresponding co-efficients of friction are W_1 W_2, μ_1, μ_2 respectively, show that the greatest inclination of the plane for equilibrium is

$$\tan^{-1}\frac{\mu_1 W_1+\mu_2 W_2}{W_1+W_2}.$$

P. U. 1940

26. Two small rings of weight W and 4W are threaded on a rough horizontal wire, the co-efficients of friction between the wire and the two rings are $\frac{3}{4}$ and $\frac{1}{2}$ respectively. The ends of a light inextensible string are attached to the rings and a weight w is suspended from the middle point of the string. The two portions of the string are each inclined at an angle of 45° to the vertical.

Prove that if the weight w is gradually increased, equilibrium is broken by sliding one of the rings and state which ring slides first. [*Ans.* ring of wt. W.

27. A light rod of rength l is supported horizontally between two rough planes, at right angles to each other, one inclined to the horizon at an angle α and the inclinations of both being greater than λ, the angle of friction. Show that the length of that portion of the rod on which a weight may be placed without disturbing equilibrium is $l \sin 2\alpha \sin 2\lambda$.

8 A ri id framework in the form of a rhombus of side a and acute angle α, rests on a rough peg whose co-efficient of friction is μ. Show that the distance between the two extreme positions which the point of contact with the peg can have is $a\mu \sin \alpha$. (*Agra 1938*)

29. A glass rod is balanced partly in and partly out of a cylindrical tumbler, with the lower end resting against the vertical side of the tumbler. If α, β be the greatest and the least angles which the rod can make with the vertical, show that the angle of friction is

$$\tfrac{1}{2}\tan^{-1}\frac{\sin^3\alpha-\sin^3\beta}{\sin^2\alpha\cos\alpha+\sin^2\beta\cos\beta}.$$

(*Agra 1936*)

30. A heavy uniform elliptic wire (axes $2a$, $2b$) is hung over a small rough peg. Show that, if the wire can be in equilibrium with any point of it in contact with the peg, the co-efficient of friction must not be less than
$(a^2-b^2)/2ab$. (*Agra 1939*)

31. A uniform ring of weight W, has a bead of weight w attached to the rim The ring is placed over a rough peg. Show that it will hang in equilibrium with any point in contact with the peg, provided the angle of friction between the peg and the ring is greater than $\sin^{-1}w/(W+w)$. (*P.U.*)

32. A cycloid is placed with its axis vertical and vertex downwards. Show that a particle can rest on it at any point which is not higher than $2a \sin^2 \lambda$ above the lowest point, where λ is the angle of friction and a is the radius of the generating circle of the cycloid. (*Agra 1941*)

CHAPTER VIII

CENTRE OF GRAVITY

8.1. Centre of Gravity. Every material body consists of an infinite number of particles each of them being attracted by the force of gravity towards the centre of the earth. If a body is small enough as compared with the size of the earth, these forces acting on the different particles are practically parallel and are proportional to the masses of the particles. The centre of these parallel forces or weights is known as the *centre of gravity* (C. G.) of the body.

8.11. C. G. is a fixed point relative to a body. If we change the position of a body, the forces of attraction at different particles still remain parallel although their direction relative to the body is not the same. But since the centre of parallel forces is independent of the direction of the forces, therefore the C. G. of the body will remain the same in whatever position the body is placed. Hence the C. G. of a body is a fixed point relative to the body.

8.12. Non-Existence of C. G. If a body is not small enough as compared to the size of the earth but is very large,

such as a mountain, then the forces due to gravity on the various particles of the body are neither parallel (being directed towards the centre of the earth,) nor are they proportional to their masses. In this case we have a system of non-coplanar and non-parallel forces, so that the system may not even compound into a single force Such a body may, therefore. have no centre of gravity.

8.13. Centre of Mass. In the case of a big body which does not possess any C. G., we introduce the idea of its centre of mass. we suppose a hypothetical system of parallel forces acting at different particles of the body and proportional to their masses. The centre of this system of parallel forces is known as the *centre of mass* (C. M.).

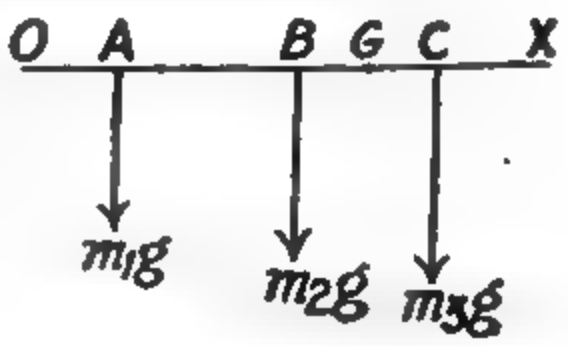
The position of the centre of mass of a body depends on the masses of its particles and their positions relative to each other.

The bodies we deal with being usually small, there is no distinction between C. G. and C. M., so that the two coincide.

Note. It is always correct to speak of the centre of gravity of a body as its centre of mass, but not *vice versa.*

8·2. Position of C. G. The following articles are usefully employed in finding the position of the C. G. of a system of particles or of bodies, the position of whose centres of gravity are separately known.

8·21. C. G. of particles placed along a straight line. Let a number of particles of masses m_1, m_2 m_3.....be placed along the straight line OX, at distances x_1, x_2, x_3,.....from a fixed point O in the line and let the resultant weight act at G at a distance $\bar{x}$ from O.



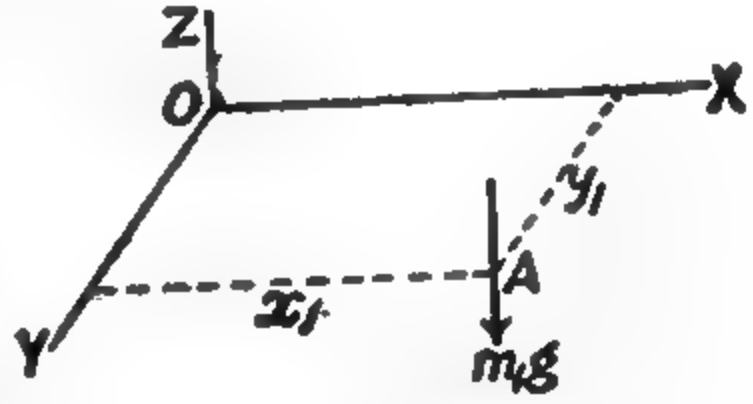
As the position of the C. G. is independent of the position of the line, we suppose, for the sake of simplicity, that the line OX is placed horizontally.

Since the moment of the resultant weight about O is equal to the sum of the moments of the separate weights, we have

$$(m_1g+m_2g+m_3g+.....)\bar{x}=m_1gx_1+m_2gx_2+m_3gx_3+.....$$

i.e. $$\bar{x} = \frac{m_1x_1+m_2x_2+m_3x_3+\cdots}{m_1+m_2+m_3+\cdots} = \frac{\Sigma mx}{\Sigma m}.$$

8·22. C. G. of particles situated on a plane. Let the particles of masses m_1, m_2, m_3,.....be placed at points A, B, C,.. .....whose co-ordinates with reference to OX, OY, perpendicular axes in the plane, are (x_1, y_1), (x_2, y_2), (x_3, y_3),.....



Since the position of the C. G. of the particles remains the same in whatever position the plane be situated, let us suppose, for the sake of simplicity, that the plane is horizontal.

Let OZ be the vertical line through O, parallel to which all the weights act and let $(\bar{x}, \bar{y})$ be the co-ordinates of the C. G. of the particles.

Then taking moments about OY, we have

$$(m_1g+m_2g+m_3g+\ldots\ldots)\bar{x} = m_1gx_1+m_2gx_2+m_3gx_3+\ldots\ldots$$

i.e. $$\bar{x} = \frac{m_1x_1+m_2x_2+m_3x_3+\ldots\ldots}{m_1+m_2+m_3+\ldots\ldots} = \frac{\Sigma mx}{\Sigma m}.$$

Similarly by taking moments about OX, we get

$$\bar{y} = \frac{m_1y_1+m_2y_2+m_3y_3+\ldots\ldots}{m_1+m_2+m_3+\ldots\ldots} = \frac{\Sigma my}{\Sigma m}.$$

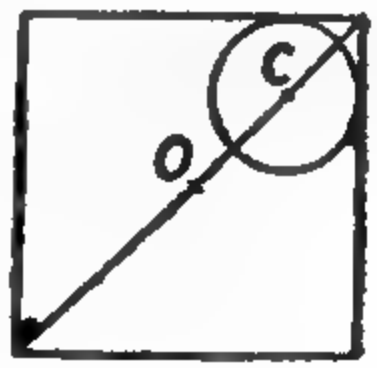
8·23. C. G. of a compound body or C. G. of a remainder. To find the C. G. of the whole when that of each of its given parts is known or to find the C. G. of the remainder when a portion whose C. G. is known has been removed, are two special cases of **8.21.** The mass of the portion removed is taken as negative.

The method is illustrated by the following examples :—

Example. (i) *A metallic uniform* circular disc of radius a is placed on a square plate of the same material and side 4a,*

*The bodies are always supposed to be of uniform density unless otherwise stated.

touching the two adjacent sides. Find the distance of the C. G. of the composite body from O, the centre of the square.



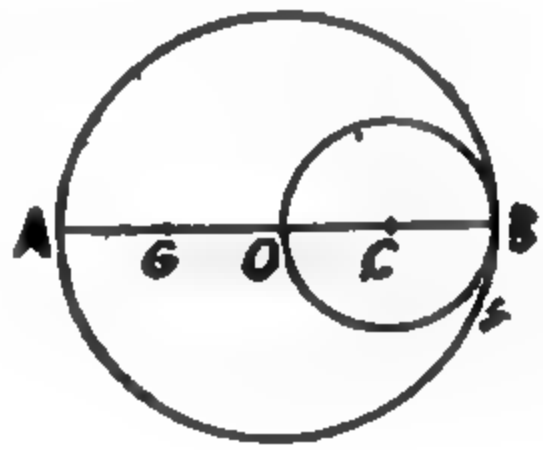
Let m be the mass per unit area. Then m_1, the mass of the square plate $=16a^2m$, acting at O.

m_2, the mass of the disc $=\pi a^2 m$ acting at C, $(OC=\sqrt{2}a)$.

Measuring distance from O along OC, we have by Art. **8·21.** the required distance

$$\bar{x}=\frac{m_1x_1+m_2x_2}{m_1+m_2}=\frac{m_1\times 0+m_2\sqrt{2}a}{m_2+m_2}$$
$$=\frac{\pi a^2m\times\sqrt{2}a}{16a^2m+\pi a^2m}=\frac{\sqrt{2}\pi a}{16+\pi}.$$

Example. (*ii*) *From a uniform circular disc of radius a, a circular hole, having a radius of the disc as diameter is punched. Find the C. G. of the remainder.*



Let m be the mass per unit area, and G the required C. G. Now m_1, the mass of the whole disc $=\pi a^2 m$ acting at O. and m_2 the mass of the portion punched

$$=\frac{\pi a^2}{4}m, \text{ acting at C.}$$

Regarding m_2 negative and measuring distances from A along AC, we have

$$x=\text{AG}=\frac{m_1x_1-m_2x_2}{m_1-m_2}=\frac{\pi a^2m\times a-\frac{1}{4}\pi a^2m\times\frac{3}{2}a}{\pi a^2m-\frac{1}{4}\pi a^2m}$$
$$=\tfrac{5}{6}a \quad \text{or GO}=\tfrac{1}{6}a.$$

The same result can also be obtained by regarding the total mass as composed of two parts $\frac{1}{4}\pi a^2m$ acting at C and the remaining $\frac{3}{4}\pi a^2m$ acting at G. Then since O is the position of their resultant, we have

$$\tfrac{3}{4}\pi a^2m\times\text{GO}=\tfrac{1}{4}\pi a^2m.\text{OC}=\tfrac{1}{4}\pi a^2m\times\tfrac{1}{2}a$$

which gives $$\text{GO}=\tfrac{1}{6}a.$$

824. *If a body is suspended freely by a string, the line joining the point of suspension and the C. G. of the body must be vertical in the line of the string.*

In this case the body is acted upon by only two forces :—

(*i*) the tension of the string acting at the point of suspension ;

(*ii*) the weight of the body acting at its C. G.

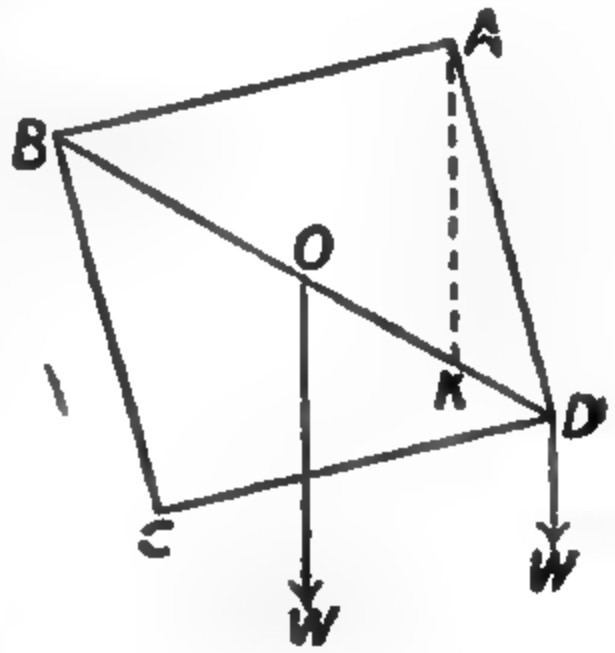
When two forces keep a body in equilibrium, they must be equal and opposite. But this cannot hold in this case unless the vertical line through the C. G. passes through the point of suspension and is in continuation of the line of the string.

Similarly it is easy to see that when a body is supported at a point and no force other than that of gravity acts on it, the line joining the point of support and the C. G. of the body is vertical.

Example. (*i*) *A square plate is suspended at one of its corners and a weight equal to its own, is suspended from an adjacent corner. Find the point in which the vertical through the point of suspension cuts the opposite diagonal of the square.*

The plate ABCD of weight W is suspended from A and a weight also equal to W is suspended from D.

The C. G. of the two weights must lie somewhere on OD, where O is the centre of the plate.



Let the vertical through A meet OD in K then since the C. G. must also lie on this vertical line, K is the C. G. of the weights.

Hence W.OK=W.KD

or OK=KD.

Hence AK divides the diagonal BD in the ratio 3 : 1.

EXAMPLES XXVIII

1. A pendulum consists of a wire of length 2ft. and weight 12 oz. to which is affixed a circular bob of wt. 2 lb., the centre of the circle being at a distance of $\frac{1}{2}$ ft. from one end of the

wire. Find the distance of its C. G. from the other end of the wire. (*Ans.* $2\frac{1}{11}$ ft.

2. The sides BC, CD of a uniform square lamina ABCD, centre O, are bisected at E and F and the square portion OECF is cut from the lamina Prove that the distance of the C. G. of the remaining portion from A is $\frac{5}{6}$OA.

3. From a body of weight W, a piece of w ight w is cut off and moved to a distance x, show that C. G. of the whole is thereby moved a distance xw/W in that direction.

4. A uniform circular lamina of radius $3a$ and centre O has a hole in the form of an equilateral triangle of side $2a$ with one vertex at O. Prove that the distance of the C. G. from O is $\frac{2a}{9\pi-\sqrt{3}}$.

5. A uniform flat circular plate of radius 2ft. has a circular hole of radius 6 in., the centre of the hole being 1ft. from the centre of the plate. Find the distance of the C. G. from the centre of the plate. [*Ans.* 0·8 in.

6 From a uniform circular disc of radius a is cut a circular hole of radius r which touches the rim of the disc. If the C. G. of the remainder is on the rim of the hole, prove that

$$r^2+ar=a^2.$$

7. A solid consists of a sphere of radius r containing a spherical cavity of radius b, the two spheres having internal contact. Find the position of the C. G. of the solid.

Show that when $b\to r$, the C. G. moves towards a point of trisection of the common diameter.

8. A uniform square board whose side is 1 ft. weighs 2 lb. and has weights 1, 2, 3, 4 lb. placed at the corners A, B, C, D respectively. Find the distances of the C. G. of the body from the sides AB and AD. [*Ans.* 8 in, 6 in.

9. Masses 5, 4, 6, 2, 7, 3 are placed at the corners A, B, C, D, E, F of a regular hexagon. Find the position of their centre of mass. [*Ans.* The centre.

10. Particles of masses p, $r+m$, q, $p+m$, r, $q+$m are placed at the angular points of a regular hexagon taken one way round. Show that the C. G. coincides with the centre of the hexagon.

11. A fine uniform wire 24 in. long is bent into the shape of a triangle ABC the sides BC, CA, AB being as 3 : 4 : 5. Particles of weights p, q, r are placed at the corners A, B and C and it is found that the C. G. is unchanged, prove that

$$p : q : r = 9 : 8 : 7.$$

12. A piece of wire 3 ft. long and of uniform linear density is bent so as to form three sides AB, BC, CD of a regular hexagon. Find the prependicular distance of the C. G. from BC.

If the wire is hung freely from A, show that in the equilibrium position AC will be vertical. [*Ans.* $2\sqrt{3}$ in.

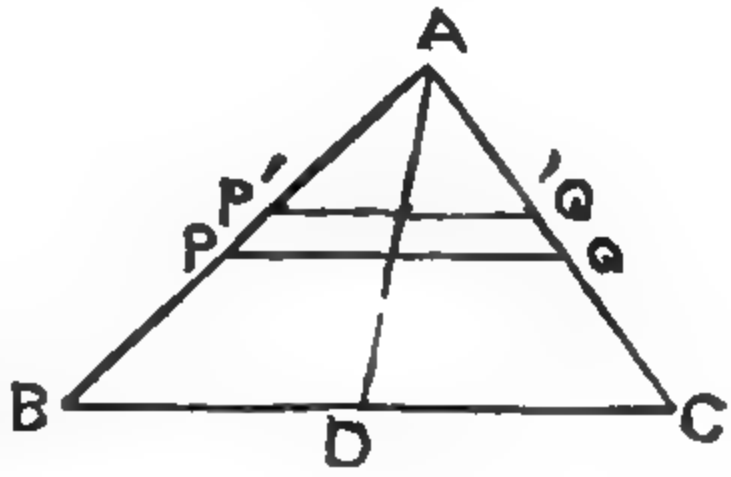
13. A light wire in the shape of an equilateral triangle ABC is suspended from A. If weights 1 and 2 lb. are respectively fastened at B and C, show that in the position of equilibrium BC makes with the horizontal an angle $\tan^{-1} \frac{1}{3}\sqrt{3}$.

14. A number of heavy particles are situated at given fixed points, and another heavy particle moves along a given straight line. Prove that the C. G. of the whole system also moves along a certain straight line.

15. A uniform rod 18 in. long, is bent so that the two parts 8 and 10 inches long are at right angles to one another. Find the distance between the centres of mass of the new shape and that of the original. (*Ali. U.*)

(*Ans.* 2·51 in.

8.3. C.G. of a triangular lamina. Consider a uniform triangular lamina ABC. Take an element of its area in the form of a strip PQQ′P′ parallel to BC. Then since the median AD bisects all straight lines paralled to BC, the centre of gravity of all strips parallel to BC lies on AD. Hence the C. G. of the whole lamina lies on AD. Similarly the C.G. of the lamina lies on the median through B or C, so that the required C. G. is the point of intersection of the medians.



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Hence C. G. of the triangular lamina is the point G on AD, one of its medians, such that AG = 2GD or AG = $\frac{2}{3}$AD.

8.31. *The C. G. of a triangular lamina is the same as that of three particles of equal mass placed at its vertices*

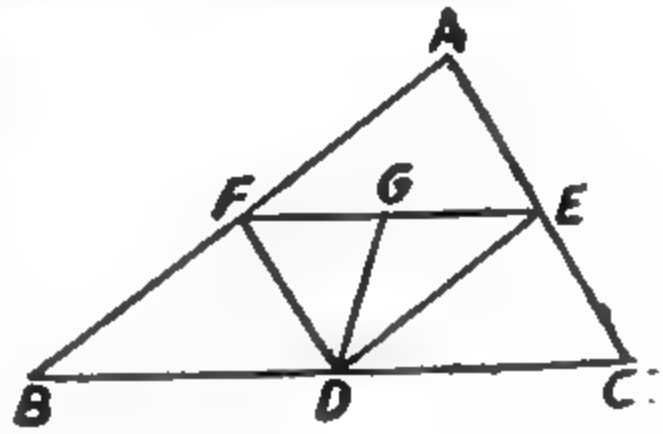
Let us consider three particles, each of mass m, placed at the vertices A, B and C. The particles at B and C are equivalent to a particle of mass $2m$ placed at D. Now the C. G. of mass m at A and $2m$ at D is some point G′ on AD such that mAG′ = $2m$ GD, *i.e.* AG′ = 2G′D. Therefore the point G′ is the same as G, the C. G. of the triangle.

In problems concerning the weight and the C. G. of a triangular lamina, the lamina can be replaced by three isolated particles each equal to one-third of the weight of the lamina, placed at the vertices.

Cor. The C. G. of a triangular lamina is the same as that of three particles of equal mass placed at the middle points of the sides.

Ex. Prove that if the C. G. of three particles placed at each of the angular points of a triangle coincides with the C. G. of the triangle, the particles have equal masses. (*D. U.*)

8.32. The C. G. of a wire in the form of a triangle. The weights of the wires forming the sides are proportional to their lengths and act at their midlle points, so that weights wb, wc of the rods AC and AB act at their middle points E and F. Their C. G. is the point G in EF such that



$$wb.\mathrm{GE} = wc.\mathrm{FG}.$$

$$\therefore \quad \frac{\mathrm{FG}}{\mathrm{GE}} = \frac{b}{c} = \frac{2\mathrm{DF}}{2\mathrm{DE}} = \frac{\mathrm{DF}}{\mathrm{DE}}$$

i.e., DG bisects the angle FDE.

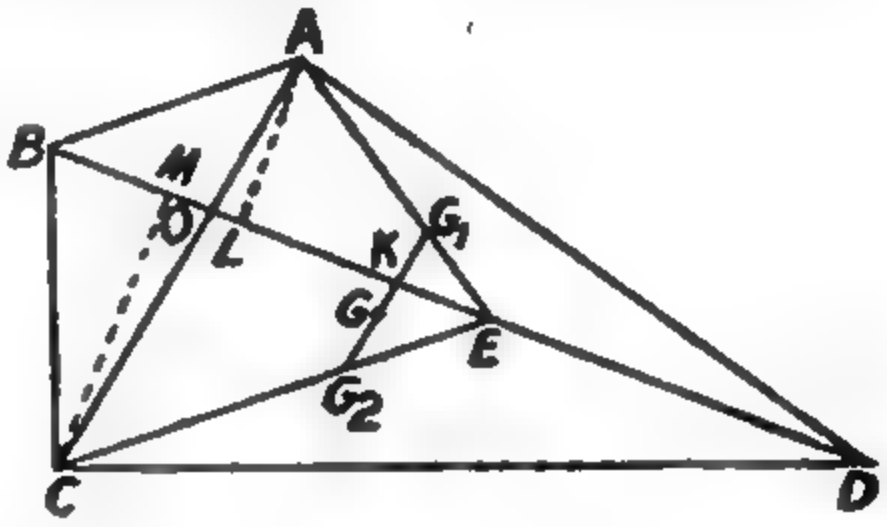
Now we have weights wa placed at D and $w(b+c)$ placed at G, therefore C. G. of the whole wire must lie on DG, the bisector of ∠EDF of the triangle DEF.

Similarly it can be shown that the C. G. lies on the internal bisector of either of the angles E or F of △ DEF.

Hence *the C. G. of the wire ABC is the incentre of the* $\triangle$ *DEF formed by joining the middle points of the sides of the* $\triangle$ *ABC.*

Ex. Show that if the weight of one of the sides is supposed to be negative, the C. G. is an excentre of the triangle formed by joining the middle points of the sides.

8.4. C. G. of a quandrilateral lamina. Let E be the middle point of the diagonal BD of the quadrilatral ARCD and G_1, G_2 the C.G's of the triangles ABD and CBD, then $G_1E=\frac{1}{3}AE$, $G_2E=\frac{1}{3}CE$ ...(*i*)



Join G_1G_2 meeting BD in K. Cut off $G_2G=G_1K$ or $G_1G=G_2K$, then G is the required C. G. of the given quadrilateral.

For, from (*i*) G_1G_2 is parallel to the diagonal AC which intersects BD in O.

$\therefore$ If AL, CM be perpendiculars drawn from A and C on BD, we have

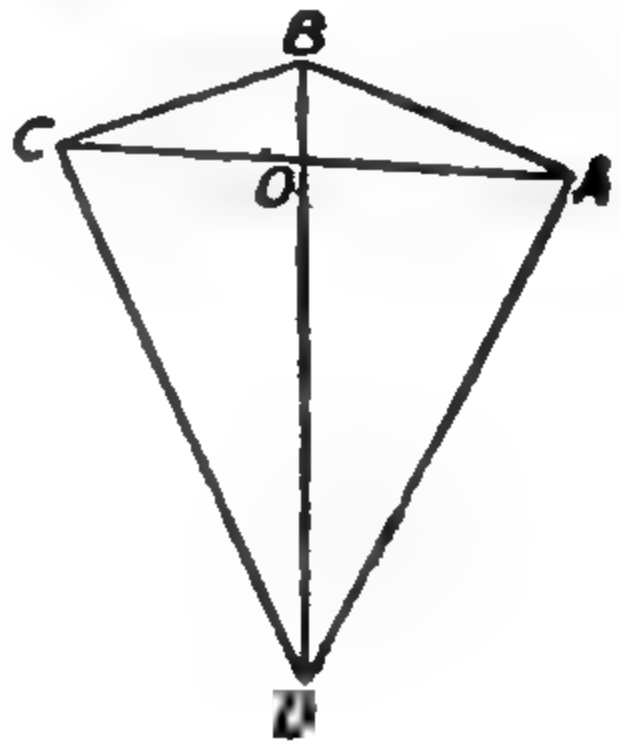
$$\frac{\triangle ABD}{\triangle CBD}=\frac{AL}{CM}=\frac{AO}{CO}=\frac{KG_1}{KG_2}=\frac{GG_2}{GG_1},$$

i.e., $\triangle ABD \times GG_1=\triangle CBD \times GG_2$,

Hence G is the C. G. of the quadrilateral.

8.41. *The C. G. of a quadrilateral lamina is the same as that of four particles of equal mass placed at the vertices and a fifth particle of equal and negative mass placed at the intersection of the diagonals.*

Let $3m_1$, $3m_2$ be the masses of the triangles ABD and CBD.



Replace $\triangle$ ABD by masses, each equal to m_1, placed at A, B and D.

Similarly replace $\triangle$ CBD by masses each equal to m_2, placed B, C, D.

Thus we have masses,

m_1 at A, m_2 at C, (m_1+m_2) at B, and (m_1+m_2) at D.

Now introduce particles m_2 at A, m_1 at C and $-(m_1+m_2)$ at O.

Since $$\frac{AO}{CO}=\frac{\triangle ABD}{\triangle CBD}=\frac{m_1}{m_2},$$

$\therefore$ O is the C. G. of mass m_2 at A and m_1 at C,

Thus the particles of masses, m_2 placed at A, m_1 placed at C and a particle of negative mass m_1+m_2 placed at O, balance.

Hence with the introduction of these additional particles the quadrilateral lamina becomes equivalent to four equal masses, each equal to m_1+m_2, say m, placed at the vertices A, B, C and D and an equal but negative mass placed at O.

8.42. Example. (*i*) *If x_1, x_2, x_3, x_4, x_5 be the distances of the four corners and the point af intersection of the diagonals of a quadrilateral from a straight line in its plane, show that the distance of the C. G. from the same line is*

$$\bar{x}=\frac{x_1+x_2+x_3+x_4-x_5}{3}.$$

(*P. U. Hons.* 1935)

Replace the quadrilateral by four particles, each of mass m placed at the corners and a fifth particle of mass $-m$ placed at O.

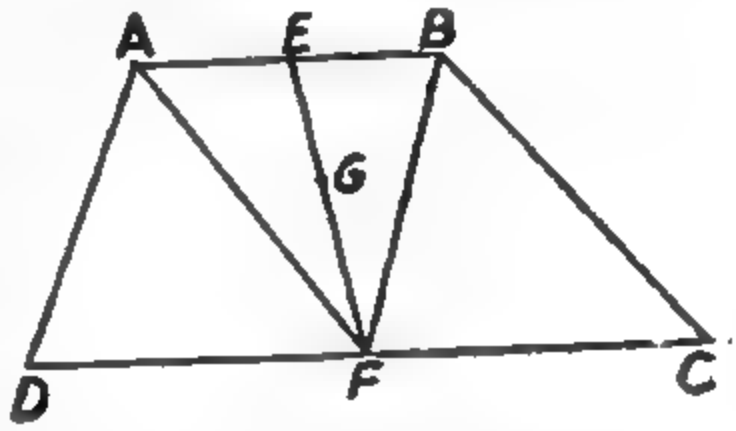
Since the position of the C. G. of a body is independent of its position, the lamina may be taken as horizontal.

Taking moments of the weights of the masses about the given line, $\bar{x}$, the distance of the C. G. from the given line is given by

$$(4m-m)\,\bar{x} = mx_1+mx_2+mx_3+mx_4-mx_5$$

which gives the result.

8.43. The C. G. of a trapezium. Let $2a$, $2b$ be the lengths and E, F the middle points of the parallel sides AB, DC of the trapezium ABCD.



The weights of the triangles ABF, ADF and BCF are proportional to $2a$, b and b respectively, let them be $2aw$, bw, and bw.

Replacing the triangles by particles equal to one-third of the weights placed at the angular points, we have $\frac{2a+b}{3}w$ at each of the points A and B ; $\frac{b}{3}w$ at each of C and D and $\frac{2a+2b}{3}w$ at F.

Again the weights at A and B are equivalent to a weight $\frac{4a+2b}{3}w$ at the middle point E and the weights at D, C and F are together equivalent to $\frac{2a+4b}{3}w$ at F.

Thus G, the C. G. of the trapezium, which is the same as that of $\frac{4a+2b}{3}w$ at E and $\frac{2a+4b}{3}w$ at F, lies on EF, such that

$$\frac{4a+2b}{3}w \times EG = \frac{2a+4b}{3}w \times GF$$

$$\therefore \quad \frac{EG}{GF} = \frac{2a+4b}{4a+2b} = \frac{a+2b}{2a+b}$$

EXAMPLES XXIX

1. The co-ordinates of the vertices of a triangle are (x_1, y_1), (x_2, y_2), (x_3, y_3), show tnat $(\bar{x}, \bar{y})$ the co-ordinates of its C. G. are given by

$$\bar{x} = \tfrac{1}{3}(x_1+x_2+x_3). \qquad \bar{y} = \tfrac{1}{3}(y_1+y_2+y_3).$$

2. Show that in the triangle ABC, right angled at C, the distance of its C. G. from a line through A perp. to AB is $\frac{1}{3}(b \cos A + b \sec A)$.

3. The side CB of the triangle ABC makes an angle θ with a fixed line CX in the plane. Show that the distance of the C. G. of the triangle from CX is

$$\tfrac{1}{3}[a \sin \theta + b \sin (\theta + C)].$$

4. Prove that the C. G. of the triangle formed by joining the middle points of the sides of the triangle is the same as the C. G. of the original triangle.

5. The C G. of a quadrilateral lies on one of its diagonals. Show that this diagonal bisects the quadrilateral.

6. If one diagonal of a quadrilateral bisects the other and its C. G. be at the middle point of the bisecting diagonal, show that the quadrilateral is a parallelogram.

7. A uniform quadrilateral plate ABCD is such that the diagonal AC bisects it and BD cuts it into two parts in the ratio of 2 : 1. Show that its C. G divides AC in the ratio of 5 : 4.

8. G is the C. G. of a uniform quadrilateral plate G′ is the C. G. of four equal particles placed at its corners, and O is the intersection of its diagonals. Prove that O, G and G′ are in the same straight line, and that

$$OG' = 3\ GG'.$$ (*P. U.* 1938)

9. If the C. G of a four-sided figure coincides with one of its angular points, show that the distances of this point and of the opposite angular point from the line joining the other two angular points are as 1 : 2. (*D. U.* 1936)

10. If the C. G. of a quadrilateral is the same as that of four equal particles placed at its angular points, show that the quadrilateral must be a parallelogram.

11. Prove that if equal triangles be cut from the corners of a given triangle by lines drawn parallel to the respective sides, the C. G. of the remainder will coincide with that of the given triangle.

12. A piece of uniform wire is bent in the shape of an isosceles triangle whose sides are a, a and b. Show that the distance of the C. G. from the base of the triangle is

$$a\sqrt{a^2-\tfrac{1}{4}b^2}/(2a+b).$$

13. A uniform wire is bent into the form of a triangle. If x, y, z are the distances of its C. G. from the three sides and r, the in-radius, show that

$$4xyz-r^2(x+y+z)=r^3.$$

14. A uniform wire is bent into the form of a triangle ABC. AD is drawn perpendicular to BC, (D being between B and C) and the part BD of the wire is cut away. Show that the distance of the C. G. of the remainder from BC is

$$c(b+c)\sin B \Big/ \left(2c+4b\cos^2\frac{C}{2}\right).$$

15. The sides AB, AC and the median AD of a triangular lamina ABC are cut in B′, C′, D′ by a line drawn parallel to BC. Show that the C. G of the quadrilateral BB′C′C is at G on DD′ and that

$$DG=\frac{l}{3}-\frac{2x^2}{3(l+x)}$$

where $AD=l$, $AD'=x$.

16. A square ABCD is divided into two parts by joining A to E, the middle point of BC. Prove that the line joining the C. G. of the triangle ABE to that of the quadrilateral AECD is perpendicular to AE.

(*P. U.* 1930.)

17. A uniform lamina is in the form of a square ABCD of side a, and E, F are the mid-points of AB, AD. The part AEF is folded about EF so that A coincides with the centre O of the square. Find the distance from O of the C. G. of the lamina in its new form.

[*Ans.* $a\sqrt{2}/48$.]

18. A triangular lamina ABC is suspended successively from A and B such that the two positions of any line in the lamina are perpendicular to one another. Prove that $a^2+b^2=5c^2$; a, b, c being the sides of the triangle. (*P. U.* 1940)

19. From a uniform square lamina ABCE a triangular portion ADE is cut away, where D is the mid-point of CE. If the lamina ABCD is freely suspended from A, show that AD will be inclined to the vertical at an angle $\tan^{-1}(\frac{4}{7})$.

20. ABCD is a lamina in the form of a trapezium with AD and BC parallel, $ABC=90°$, $AB=a$, $BC=2a$, $AD=a$. The lamina is suspended by a string attached to the vertex B. Show that the inclination of BC to the vertical in the position of equilibrium is $\tan^{-1}(\frac{4}{7})$.

21. Prove the following construction for the C. G. of a trapezium ABCD of which the sides AB, DC are parallel :—

Produce AB to E making BE = DC, and CD to F making DF = AB. Then the C. G. is the point of intersection of EF with the line joining the middle points of AB and CD.

22. Prove that the C. G. of a uniform triangular lamina of mass M, bordered with a thin uniform rim of mass m, and loaded with a particle of mass $\frac{1}{2}m$ at the centre of its inscribed circle, is at the centroid of the triangle.

23. From a triangular board of uniform thickness, the portion constituting the area of the inscribed circle is removed, prove that the distance of the C. G. of the remainder from the side a is

$$\frac{\triangle}{3as}\cdot\frac{2s^3-3\pi a\triangle}{s^2-\pi\triangle}$$

where $\triangle$ is the area and s, the semi-perimeter of the board.

8.5. Determination of C. G. by Intergation. We have seen in 8.22 that in the case of a system of particles of masses m_1, m_2, m_3.....placed at (x_1, y_1), (x_2, y_2), (x_3, y_3),..... the co-ordinates of the C. G. $(\bar{x}, \bar{y})$ are given by

$$\bar{x}=\frac{\Sigma m x}{\Sigma m},\quad \bar{y}=\frac{\Sigma m y}{\Sigma m},$$

If instead of a system of isolated particles distributed at separate points, we are to deal with a continuous distribution of mass, then the position of the C. G. is calculated with the help of integration as illustrated by the following articles.

8.51. C G. of a rod

(*i*) Let the rod be *uniform*, its mass per unit length being m and let $2a$ be its length.

Let $PQ=\delta x$ be an element of the rod, then the mass of this element $=m\delta x$ and the distance from A of the C. G of the element is approximately $=x$.

$\therefore$ $\bar{x}$, the distance of C. G. of the rod from A

$$=\frac{\int_0^{2a} x.mdx}{\int_0^{2a} mdx}=\frac{2a^2}{2a}=a.$$

(*ii*) Let the rod be of *variable density*, the density varying as the distance from one end A.

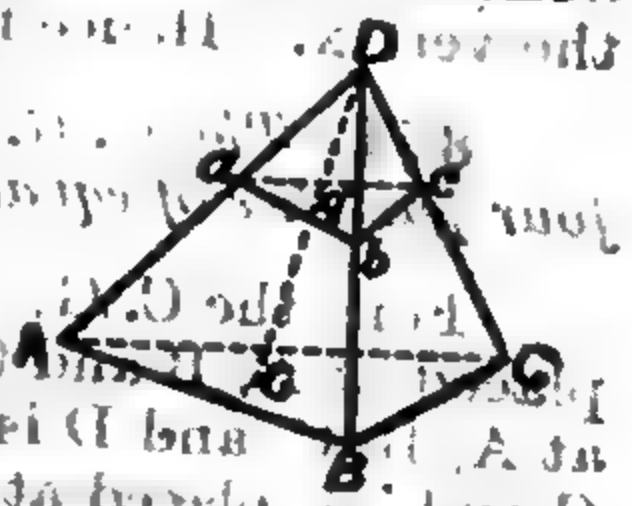
Here the density $\rho=kx$ and the mass of the element $=kx\delta x$

$$\therefore \bar{x}=\frac{\int_0^{2a} x.kxdx}{\int_0^{2a} kxdx}=\frac{\int_0^{2a} x^2dx}{\int_0^{2a} xdx}=\tfrac{4}{3}a=\tfrac{2}{3}.2a$$

8.52. C. G. of a tetrahedron. Let us consider an element of the tetrahedron in the form of a triangular lamina abc parallel to the base ABC of the tetrahedron ABCD.

Join the vertex D with G, the C. G. of $\triangle ABC$, meeting $\triangle abc$ in q.

Then by geometry of similar triangles it can easily be shown that q is the C. G. of $\triangle abc$.



$\therefore$ the mass of the lamina abc can be replaced by the mass of a single particle at q.

Also by geometry, the area or the mass of the lamina abc is proportional to the square of the distance Dq.

Hence by dividing the tetrahedron into laminae such as abc, we can regard the tetrahedron equivalent to the rod $DG(=b)$, its density varying as the square of the distance from D.

Hence $\bar{x}$, the distance of the C. G. of the tetrahedron from D along DG

$$= \frac{\int x\, m\,dx}{\int m\,dx} = \frac{\int_0^b x.kx^2dx}{\int_0^b kx^2dx} = \tfrac{3}{4}b.$$

Hence the C. G. of a tetrahedron lies on the line joining one of it vertices to the centroid of the opposite face and divides it in the ratio 3 : 1 *from the vertex.*

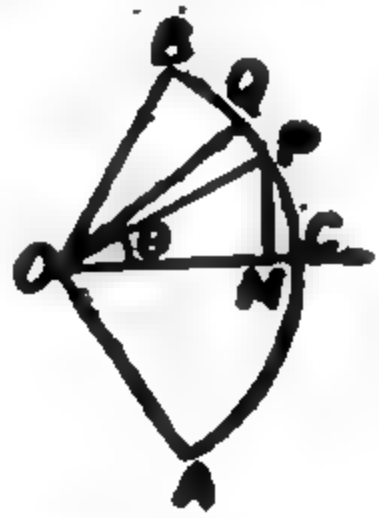
8.53. C. G. of a pyramid. The above proof is also applicable to a pyramid with its base in the form of a polygon. The C. G. in this case lies on the line joining the vertex D to G, the C.G. of the base and is situated at a distance $\frac{3}{4}$DG from D.

Example. Show that in finding the C. G. of a triangular lamina, the lamina can be regarded as equivalent to a rod along a median with its density varying as the distance from the vertex. Hence find its C. G.

8.54 *The C. G. of a tetrahedron is the same as the C. G. of four particles of equal mass placed at its vertices.*

For, the C. G. of three equal particles, each of mass m, placed at A, B and C is the point G, and of the four particles at A, B, C and D is the same as that of masses $3m$ placed at G and m placed at D which is a point on DG dividing DG in the ratio 3 : 1 from D.

8.6. C. G. of a uniform circular wire. Let the wire in the form of an arc AB of a circle of radius a and centre O subtend an angle 2α (in radians) at O. Let C be its middle point.



Then, by symmetry, the C. G. of the wire will lie on OC.

Let $\angle COP=\theta$, $\angle COQ=\theta+\delta\theta$, and let m be the mass per unit length.

Then the mass of the element PQ of the wire $=ma\delta\theta$, and the distance of the C. G. of this element from O along OC is approximately $=a\cos\theta$.

$\therefore \bar{x}$, the distance of the C. G. of the whole arc

$$\text{from O along OC}=\frac{\int_{-\alpha}^{\alpha} a\cos\theta \cdot mad\theta}{\int_{-\alpha}^{\alpha} mad\theta}=\frac{a\sin\alpha}{\alpha}.$$

In the particular case of a semi-circular arc, α being $\frac{\pi}{2}$,

$$\bar{x}=\frac{a}{\frac{\pi}{2}}=\frac{2a}{\pi}.$$

Example. *Find the C. G. of a bow, consisting of an arc AB of a circle of radius a subtending an angle 2α at the centre O, and its chord, formed by a uniform wire.*

By symmetry, the C. G. of the chord, the arc and the whole bow lies on OC, C being the middle point of the arc AB.

Let m be the mass per unit length of the wire, then

m_1, the mass of the chord $=m.2a\sin\alpha$.

x_1, the distance of its C. G. from O $=a\cos\alpha$.

m_2, the mass of the arc $=m.\ a.\ 2\alpha=2ma\alpha$

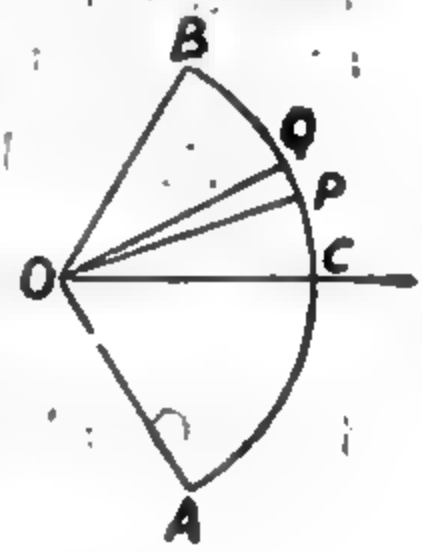
x_2, the distance of its C. G. from O $=(a\sin\alpha)/\alpha$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$= \frac{2am \sin a . a \cos a + 2ama . a \sin a/a}{2am \sin a + 2ama}$$

$$= \frac{a \sin a (1 + \cos a)}{\sin a + a}.$$

8.61. C. G. of a Sector of a Circle. Let C be the sector of radius a and centre O. Let C be the middle point of the arc AB and let $\angle AOB = 2a$ (in radians), then by symmetry the C. G. of the sector must lie on OC.



Let m be the mass per unit area and let $\angle COP = \theta$, then the mass of the element OPQ of the sector $= m . \frac{1}{2} a^2 \delta\theta$. Taking the sectorial element OPQ to be approximately equal to $\triangle OPQ$, the distance of the C. G. of this element from O along OC $= \frac{2}{3} a \cos\theta$.

$\therefore$ $\bar{x}$ = the distance of the C. G. from O along OC.

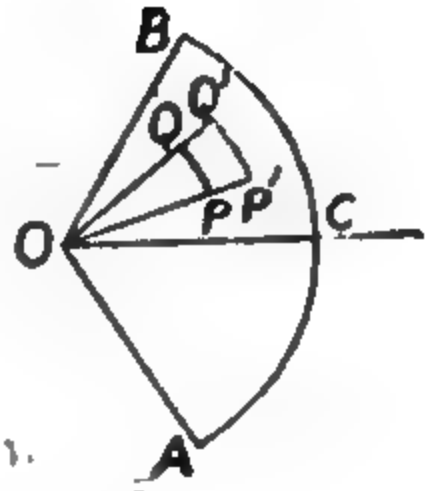
$$= \frac{\int_{-a}^{a} \frac{2}{3} a \cos\theta . \, m \frac{1}{2} a^2 d\theta}{\int_{-a}^{a} m \, \frac{1}{2} a^2 d\theta} = \frac{2}{3} a \frac{\sin a}{a}.$$

Cor. *In the case of a semi-circular lamina* $a = \frac{\pi}{2}$ *and*

$$\bar{x} = \frac{4a}{3\pi}.$$

8·611 The same result can also be obtained by the use of double integration as follows :—

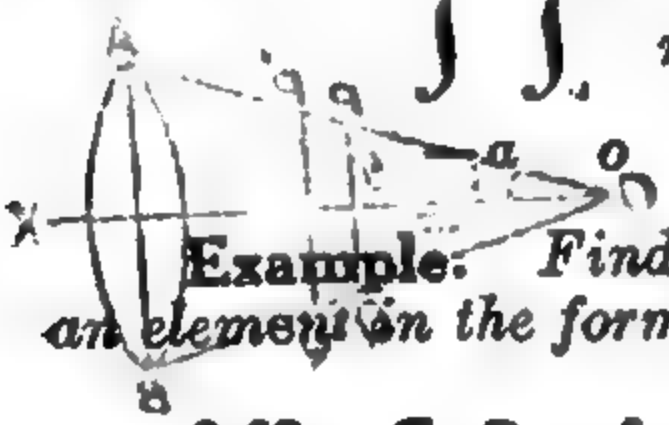
Co-ordinates of P (r, θ), and of $Q'(r+\delta r, \theta+\delta\theta)$.



Hence mass of the element PP'Q'Q
$= m\delta r . r\delta\theta$.

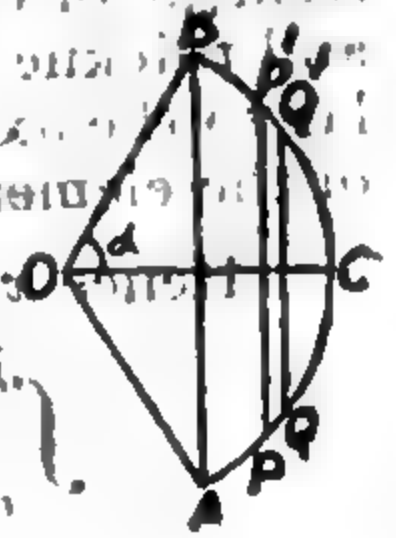
and the distance of its C. G. from O along
$OC = r\cos\theta$.

$$\text{Hence } \bar{x} = \frac{\int_{-a}^{a}\int_{0}^{a} m dr . r d\theta\, r\cos\theta}{\int_{-a}^{a}\int_{0}^{a} m\, dr\, r d\theta} = \frac{\int_{-a}^{a} \frac{a^3}{3}\cos\theta\, d\theta}{\int_{-a}^{a} \frac{a^2}{2} d\theta} = \frac{2}{3}\frac{a\sin a}{a}$$



Example. *Find the C. G. of a sector of a circle by taking an element in the form of a strip bounded by radii r and $r+\delta r$.*

8.62. C. G. of a Segment of a circle. Let O be the centre and C the middle point of the arc AB of the segment ACB.



Taking the element in the form of a strip PQQ'P' parallel to the chord AB at a distance x from O, we have the mass of the element $= m2y\delta x$ where m is the mass per unit area and $x^2+y^2=a^2$

$\therefore$ $\bar{x}$, the distance of C. G. from O along OC

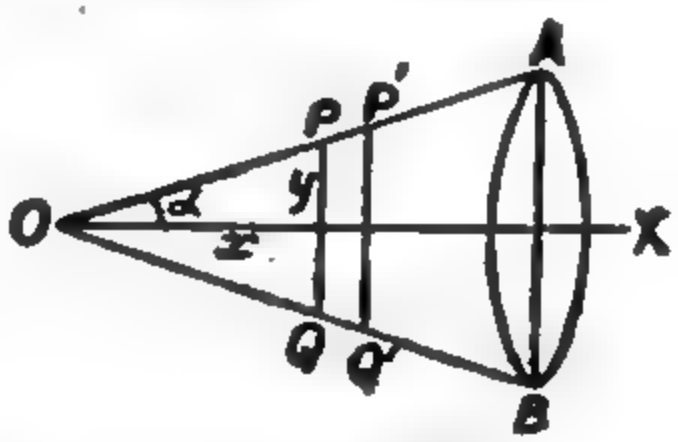
$$= \frac{\int_{a\cos a}^{a} x.2y\, dx}{\int_{a\cos a}^{a} 2y\, dx} = \frac{\int_{a\cos a}^{a} x\sqrt{a^2-x^2}\, dx}{\int_{a\cos a}^{a} \sqrt{a^2-x^2}\, dx}$$

$$= \frac{\begin{vmatrix} -\frac{1}{3}(a^2-x^2)^{\frac{3}{2}} & a \\ & a\cos\alpha \end{vmatrix}}{\begin{vmatrix} \frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\frac{x}{a} & a \\ & a\cos\alpha \end{vmatrix}}$$

$$= \frac{2a}{3}\cdot\frac{\sin^3\alpha}{\alpha-\sin\alpha\cos\alpha}.$$

The same result can also be obtained by regarding the segment as the difference of the sector OACB and the triangle OAB.

87. C. G. of a right circular cone. Let α be the semi-vertical angle, O, the vertex and h, the height of the cone.



Taking the axis of the cone as the axis of x, the equation of the straight line OA is $y = x\tan\alpha$.

(i) **Solid cone.** Let us take an element of the cone in the form of a disc PP'Q'Q, of radius y and thickness δx, situated at a distance x from O, perpendicular to the axis. If m is the mass per unit volume, the mass of the element $= \pi y^2 \delta x . m$

Hence $\bar{x}$ the distance of the C. G. from O along Ox.

$$= \frac{\int_o^h x.\pi y^2 dx}{\int_o^h \pi y^2 dx} = \frac{\int_o^h x.^2x\tan^2\alpha\, dx}{\int_o^h x^2\tan^2\alpha\, dx} = \tfrac{3}{4}h$$

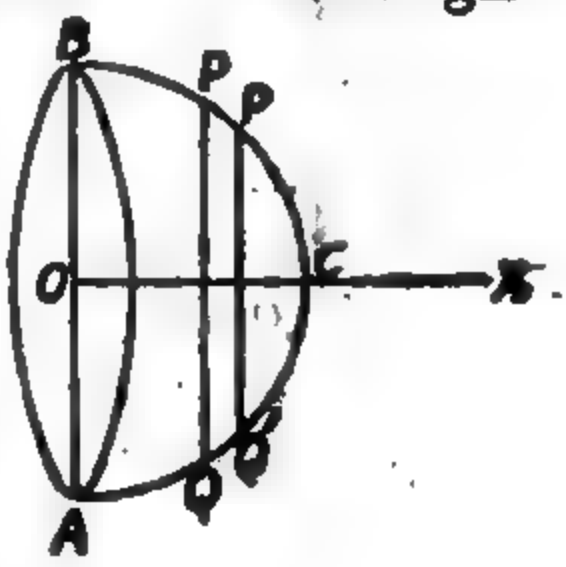
(ii) **Hollow cone.** Here we take the element in the form of a belt PP'Q'Q on the surface of the cone.

Width of the element $= PP' = \delta x \sec\alpha$.

If m is the mass per unit area, the mass of the element $= m.2\pi y\ \delta x \sec\alpha$.

$$\text{Hence} \quad \bar{x}=\frac{\int_0^h x.2\pi my \sec\alpha\, dx}{\int_0^h 2\pi my \sec dx}=\frac{\int_0^h x.x\tan\alpha\, dx}{\int_0^h x\tan\alpha.\, dx}=\tfrac{2}{3}h.$$

871. C. G. of a hemisphere. Let OC, the line through the centre O, perpendicular to the plane AOB be the axis of x. Equation of the circle ACB is $x^2+y^2=a^2$.

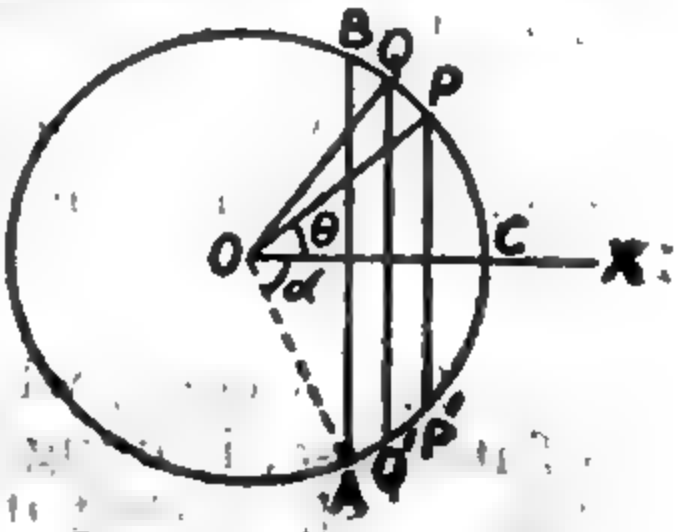


Take an element in the form of a disc, PP'Q'Q of radius y and thickness δx, situated parallel to the base at a distance x from it. If m be the mass per unit volume, mass of the element $=m\pi y^2\,\delta x$.

$$\therefore\quad \bar{x}=\frac{\int_0^a x.m\pi y^2\,dx}{\int_0^a m\pi y^2\,dx}=\frac{\int_0^a x(a^2-x^2)\,dx}{\int_0^a (a^2-x^2)\,dx}=\tfrac{3}{8}a.$$

Cor. *For C. G. of a segment of a sphere of height $\frac{1}{2}a$, the limits of integration are from $\frac{1}{2}a$ to a and* $\bar{x}=\frac{27}{40}a$.

8.72. C. G. of a spherical cap. Let the plane AB (perpendicular to the plane of the paper) cut a spherical cap ACB. Let OX, perpendicular to the base AB meet the curved surface in C. Let us take PP'Q'Q, an element of the surface parallel to the base AB.



Let $\angle COP=\theta$, $\angle COQ=\theta+\delta\theta$, then the arc $PQ=a\delta\theta$. If m be the mass per unit area, the mass of the belt $=m2\pi\, a\sin\theta\; ad\theta$.

If $\angle AOB=2\alpha$, the angle θ varies from 0 to α. Hence $\bar{x}$, the distance of the C. G. from O along OC

$$= \frac{\int_0^a a\cos\theta . m 2\pi a \sin\theta \, a d\theta}{\int_0^a m . 2\pi a \sin\theta \, a d\theta}$$

$$= a \frac{\int_0^a \sin\theta\cos\theta \, d\theta}{\int_0^a \sin\theta \, d\theta} = \frac{a}{2} \cdot \frac{1-\cos^2\alpha}{1-\cos\alpha} = \frac{a+a\cos\alpha}{2}$$

i.e., the C. G. bisects the axis of the cap ; the same result can be obtained by using cartesian co-ordinates.

In the particular case of a hollow hemisphere, α being $\frac{\pi}{2}$,

$$x = \frac{a}{2}.$$

Cor. *The C. G. of spherical zone lies midway between its two parallel bounding planes.*

EXAMPLES XXX

1. A uniform wire is bent into the form of a circular arc and two bounding radii and the C. G. of the whole is found to be at the centre. Show that the angle subtended by the arc at the centre is $\tan^{-1}(-\frac{4}{3})$.

2. If a sector of a circle balance about the chord of the arc, prove that the half angle α of the sector is determined by

$$2\tan\alpha = 3\alpha.$$

3. A cone, whose height is equal to four times the radius of its base, is hung from a point in the circumference of its base. Show that it will rest with its base and axis equally inclined to the vertical. (*P. U.* 1937)

4. A solid of uniform density is built up of a hemisphere of radius r, and a right circular cylinder of radius r and height

h on the circular base of the hemisphere. Find the position of the C. G. of the solid.

[Ans. At a distance $\frac{3}{4}\cdot\frac{2h^2-r^2}{3h+2r}$ from the common base.

5. A hollow right circular cone, without base, of mass M, and made of thin metalic sheet is placed with its axis vertical and vertex downwards. Water is poured into it up to a depth of one-half of the vertical height Prove that, if the C. G. of the coue and its contents is at the surface of the water, the mass of water is $\frac{4}{5}$M.

6. From a semi-circular lamina of radius $2a$, a circular lamina of radius a is removed. Find the position of the C G. of the remainder. (*D. U.* 1940)

[*Ans.* At a distance $\frac{16a}{3\pi}-a$ from the diameter.

7. A plane lamina is limited by a semi-circle whose diameter is AB ($=2a$) and two semi circles whose diameters are AO and OB, O being the centre of the semi-circle AB. Find the C. G. of any one of the three figures so formed.

[*Ans.* $\left(\frac{2a}{3\pi}, 0\right)$, $\left(\frac{2a}{\pi}, 0\right)$, $\left(\frac{a}{\pi}, \frac{1}{4}a\right)$

8. Any point P is taken upon the diameter AB of a semi-circular area and the semi-circles upon AP BP as diameter are removed. Find the C G of the area that is left.

[*Ans.* With centre as origin and AB as x-axis, $x=b-\frac{1}{3}a$, $y=2a/\pi$ where AP$=2b$ and AB$=2a$].

9. ABC is a triangle inscribed in a circle whose centre is O, and G_1, G_2, G_3 are the C. G's of the sectors AOB, BOC, and COA respectively, prove that

$$\frac{AB}{OG_1}+\frac{BC}{OG_2}+\frac{CA}{OG_3}=3\pi.$$

10. Prove that the distance from O of G, the C. G. of a quadrant of a circle centre O, radius r, is $4r\sqrt{2/3\pi}$.

Find the position of the C. G. of the remainder when a quadrant is removed from a uniform circular lamina of radius r.

[*Ans.* distant $4r\sqrt{2/3\pi}$ from the centre,

11. An inkwell consists of a cube of glass of edge $4a$, with a hemispherical hollow of radius a, the centre of the curved surface is in one face of the cube. Find the height of the C. G. of the inkwell above the base. (Take $\pi = \frac{22}{7}$). (*D. U.* 1942)

[*Ans.* $1\frac{189}{200}a$.

12. Find the C. G. of the surface and the volume of the part of a uniform sphere of radius a, included between two parallel planes at distances b and c from its centre.

(*P. U. Hons.* 1940)

[*Ans.* $\frac{b+c}{2}$, $\frac{3}{4}$, $\frac{(b+c)(2a^2-b^2-c^2)}{3a^2-(b^2+bc+c^2)}$

13. A uniform sphere of radius a, is divided into two parts by a plane distant c from is centre. Show that the distance between the centres of gravity of the two parts is $3a^3/(4a^2-c^2)$.

What happens when $c=0$ and when $c=a$?

14. From a right circular cone of height h, a conical portion is removed of height h' and with the same base. If the C. G. of the remainder is at the vertex of the conical hollow, prove that $h=3h'$.

15. A uniform rod of length l with its cross-section a circle is sharpened at one end like a pencil, to a conical point without shortening the axis of the cylinder. If the height of this cone is h, show that the C. G. has been displaced by a distance

$$\frac{h}{4}\cdot\frac{4l-3h}{3l-2h}.$$

(*D. U.* 1940)

16. Two tangents are drawn to a circle of radius a, subtending an angle 2α at the centre. Prove that the C. G. of the figure bounded by the tangents and the smaller arc between them is at a distance from the centre equal to

$$\frac{a}{3}\cdot\frac{\tan^2\alpha\sin\alpha}{\tan\alpha-\alpha}.$$

17. Distinguish between centre of parallel forces, centre of gravity, and centre of mass.

A uniform bowl has for its inner and outer surface concentric hemispheres of radii a and b; prove that the distance of its C. G. from the geomatric centre is

$$\frac{3(a+b)(a^2+b^2)}{8(a^2+ab+b^2)}.$$ (*Agra* 1940)

18. A thin uniform rod is bent at one end to form a walking stick with a semi-circular handle. The straight portion AB is of length $2l$ and the portion BC is of radius a (less than l). The straight line joining the ends B and C of the semi-circular portion is perpendicular to AB.

The stick hangs freely with the end C supported on a horizontal table. Prove that the straight part makes an angle θ with the vertical where

$$\tan\theta = \frac{a(\pi a+4l)}{2(l^2-a^2)}.$$ (*D. U.* 1941)

19. The radii of the ends of a frustum of a uniform right circular cone of vertical angle 2α are a and $3a$. If the frustum can rest in equilibrium with its curved surface in contact with a horizontal plane, show that

$$30\cos^2\alpha > 13.$$

20. From a hollow right circular cone of semi-vertical angle α, a frustum is cut off by two planes perpendicular to the axis. If A and B are the centres and a and b the radii of the plane ends and G the C. G. of the curved surface of the frustum, prove that

(*i*) G divides the axis AB in the ratio

$$a+2b : b+2a.$$

(*ii*) the distance of G from the plane midway between the ends is

$$(a-b)^2\cot\alpha \,/\, 6(a+b).$$

Miscellaneous Examples IV.

1. Prove that the resultant of co-planar forces, represented by $\lambda_1 OA_1, \lambda_2 OA_2 \ldots\ldots \lambda_n OA_n$ is represented by $(\lambda_1+\lambda_2+\ldots+\lambda_n)OG$ where G is the C. G. of the masses λ_1 at A_1 λ_2 at $A_2 \ldots\ldots \lambda_n$ at A_n.

Two co-planar forces are represented by $\lambda AA'$, $\mu BB'$. Prove that if G, G′ are the centres of gravity of masses λ, μ at A, B and of masses λ, μ at A′, B′, their resultant is represented in magnitude and direction, but not, in general, in line of action by $(\lambda+\mu)GG'$. (*L. U.*)

2. If $\lambda_1+\lambda_2+\lambda_3 \neq 0$, show that the resultant of forces $\lambda_1 OA$, $\lambda_2 OB$, $\lambda_3 OC$ along the sides OA, OB, OC of a tetrahedron OABC is $(\lambda_1+\lambda_2+\lambda_3)$ OD along OD, where D is the centroid of particles of masses λ_1, λ_2, λ_3 at A, B, C respectively.

Deduce that, if in addition forces $l_1 BC$, $l_2 CA$, $l_3 AB$ act along BC, CA, AB respectively, the resultants of the two sets intersect if $l_1\lambda_1+l_2\lambda_2+l_3\lambda_3=0$

3. Prove that the distance of the C. G. of a uniform cube of edge a from one corner of the cube is $a\sqrt{3}/2$.

Eight equal uniform cubes of edge a are placed together to form a larger cube of edge $2a$. If one of the small cubes is removed, prove that the distance from the centre of the large cube of the C. G. of the remainder is $a\sqrt{3}/14$.

4. $(n+1)$ bricks of the same size are piled one above another in a vertical plane so that they rest, each one overlapping the one below by as much as possible.

Prove that if $2a$ is the length of each brick, the lowest but one overlaps the lowest by a length a/n.

Show also that if each brick overlaps the one next below by a length a/n, the greatest number of bricks that can be piled up is $2n-1$. (*P. U.* 1927)

5. A uniform square lamina ABCD of side a and centre G, can turn freely in a vertical plane about a fixed point at C. It rests with the point B on a smooth horizontal table, with A lower than C and CA inclined to the horizontal at $\cot^{-1} 4$. Prove that the reaction at C is not altered when the portion OCD is removed.

6. A uniform prism whose cross-section is an isosceles triangle ABC right angled at A, rests with the plane face containing AC, on a horizontal table. D is the middle point of AB and E is a point in AC distant x from A. The prism

is cut by a plane passing through DE parallel to its edges, and the portion ADE is removed. Prove that the remaining part will topple over if x is greater than $(3-\sqrt{5})a$ where $AC=2a$.

7. A frustum of a right circular cone of vertical angle 2α is obtained by cutting the cone by a plane parallel to the base at a distance h/n from the vertex, h being the height of the complete cone.

If the frustum is placed with its curved surface in contact with a horizontal table, prove that it will not topple over if

$$\cos^2\alpha > \frac{4(n^3-1)}{3(n^4-1)}.$$

8. A thin uniform rod of length l is bent into the form of a circular are whose radius a is large compared to l. Prove that the displacement of the C. G. is $l^2/24a$ approximately.

(*All. U.*)

9. ABCD is a square of side $2a$. A circle is drawn with centre A and radius $2a$. Find the C. G. of the portion of the area of the square which is not within this circle.

[*Ans.* $4\sqrt{2}a/3(4-\pi)$ from A along AC

10. Show how to cut off from a uniform cylinder a cone, whose base coincides with that of the cylinder, so that the C. G. of the remaining solid may coincide with the vertex of the cone. [*Ans.* Ratio of their heights $=2-\sqrt{2}$. (*Bn. U.*)

11. Find the vertical angle of the cone in order that C.G. of its whole surface, including its plane base may coincide with the C. G. of the volume. [*Ans.* $2\sin^{-1}(1/3)$

12. A hollow conical vessel, of perp. height h and slant height l made of thin sheet of metal is closed at its base of radius r. It is cut across by a plane parallel to its base at half the perp. height, and the upper cone is removed. Prove that the distance from the base of the C.G. of the remainder is

$$2lh/3(3l+4r).$$

13. Perpendiculars are drawn from the angular points A, B, C of a triangle to the opposite sides a, b, c and another triangle is formed by joining the feet of these perpendiculars; if x, y, z

be the distances of the C. G. of this triangle from the sides a, b, c, prove that

$$\frac{x}{a^2\cos(B-C)}=\frac{y}{b^2\cos(C-A)}=\frac{z}{c^2\cos(A-B)}.$$

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14. The bisectors of the angles of a triangle of sides a, b, c and area Δ, meet the opposite sides in D, E and F. Prove that the distance of the C. G. of the triangle DEF from the side a is

$$2(2a+b+c)\Delta/3(a+b)(a+c).$$

8.8. C. G. when density varies. When the density of a body varies according to some law of variation, it is expressed in terms of the variable or variables involved in the integrals determining the C. G. of the body.

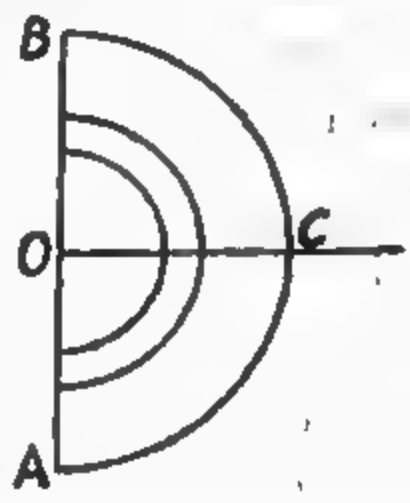
Example. *Find the C. G. of a semi-circular area when the density :—*

(*a*) *varies as the square of the distance from the centre ;*

(*b*) *varies as the square of the distance from the diameter ;*

(*c*) *varies as the distance from one end of the diameter.*

(*a*) The density being a function of the distance from the centre, we take element in the from of a semi-circular ring bounded by radii r and $r+\delta r$ then ρ, the density of the element $=kr^2$,



and mass of the element $=\rho\ \pi r\,dr$.

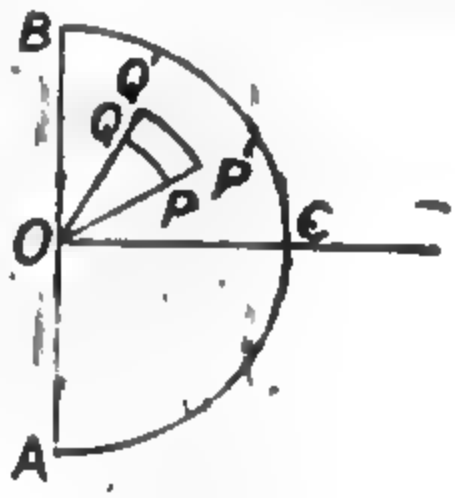
Distance of the C. G. of this element from O along $OC=\frac{2r}{\pi}$.

$$\therefore \bar{x} = \frac{\int_0^a \frac{2r}{\pi}\,\rho\pi r\,dr}{\int_0^a \rho\pi r\,dr} = \frac{\int_0^a 2r^2\,kr^2 dr}{\int_0^a \pi r.kr^2\,dr}$$

$$= \frac{2}{\pi}\cdot\frac{\int_0^a r^4 dr}{\int_0^a r^3 dr} = \frac{8a}{5\pi}.$$

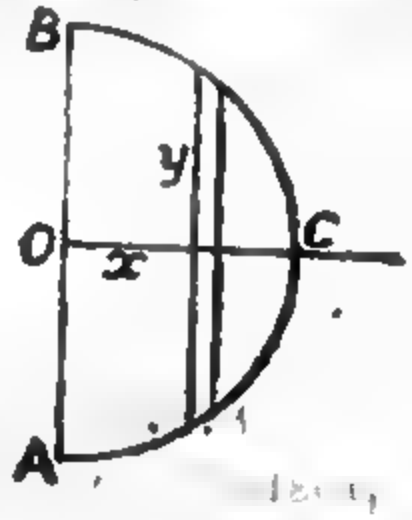
Aliter. Taking the element in the from $PP'Q'Q = r\delta\theta\delta r$

$$\bar{x} = \frac{\int_0^a \int_{-\pi/2}^{\pi/2} r\cos\theta.\rho\, r d\theta dr}{\int_0^a \int_{-\pi/2}^{\pi/2} \rho\, r d\theta\, dr} \quad \text{where} \quad \rho = kr^2$$



$$= \frac{\frac{1}{5}a^5 \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta}{\frac{1}{4}a^4 \int_{-\pi/2}^{\pi/2} d\theta} = \frac{8a}{5\pi}$$

(*b*) In this case the density being a function of the distance from the diameter, we take an element in the form of a strip parallel to the diameter at a distance x from it.



Here $\rho = ka^2$, and mass of the element $= kx^2.2ydx$.

$$\therefore x = \frac{\int_0^a x.kx^2\ 2ydx}{\int_0^a kx^2 2ydx} = \frac{\int_0^a x^3\sqrt{a^2-x^2}\,dx}{\int_0^a x^2\sqrt{a^2-x^2}\,dx}$$

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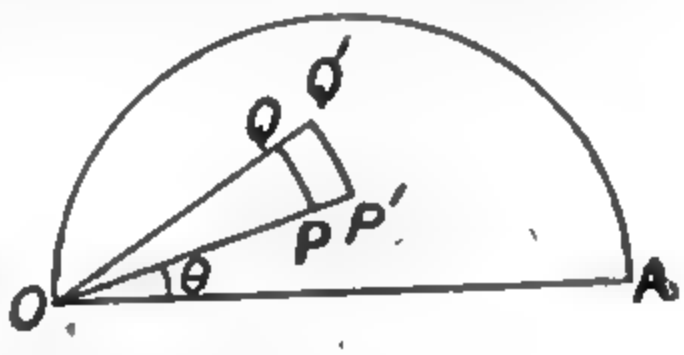
Putting $x=a\cos\theta$, we have

$$\bar{x}=a\cdot\frac{\int_o^{\pi/2}\cos^3\theta\sin^2\theta\,d\theta}{\int_o^{\pi/2}\cos^2\theta\sin^2\theta\,d\theta}=\frac{32a}{15\pi}.$$

(c) Taking OA as the initial line, the equation of the circle is $r=2a\cos\theta$.

For the element PP'Q'Q, if OP$=r$, then $\rho=kr$ and the mass$=kr\,r\,d\theta\,dr$.



Since there is no symmetry, we find both x and y for the C. G.

$$\bar{x}=\frac{\int_o^{\pi/2}\int_o^{2a\cos\theta} r\cos\theta\cdot kr^2\,d\theta\,dr}{\int_o^{\pi/2}\int_o^{2a\cos\theta} kr^2\,d\theta\,dr}=\frac{3a}{2}\frac{\int_o^{\pi/2}\cos^5\theta\,d\theta}{\int_o^{\pi/2}\cos^3\theta\,d\theta}=\frac{6a}{5}$$

$$\bar{y}=\frac{\int_o^{\pi/2}\int_o^{2a\cos\theta} r\sin\theta\cdot kr^2\,d\theta\,dr}{\int_o^{\pi/2}\int_o^{2a\cos\theta} kr^2\,d\theta\,dr}=\frac{3a}{2}\frac{\int_o^{\pi/2}\cos^4\theta\sin\theta\,d\theta}{\int_o^{\pi/2}\cos^3\theta\,d\theta}$$

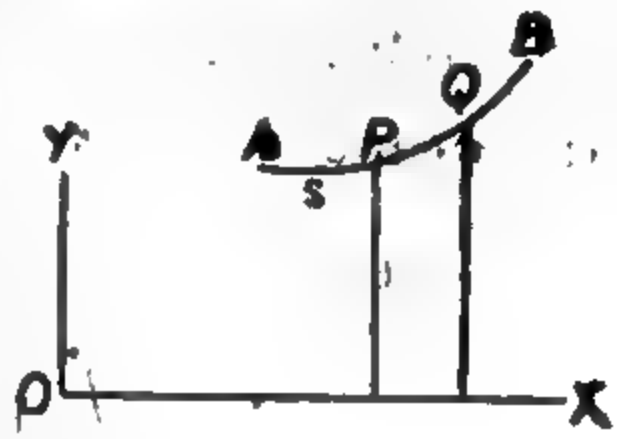
$$=\frac{9a}{20}.$$

Centre of Gravity in general.

8.9. We now deal with general methods of finding the position of C. G. of a wire, a plane area and a surface and volume of revolution.

8.91. C. G. of a wire in the form of a plane curve.

Let ρ be the mass per unit length of the wire APQB. Measuring the length of the wire from a fixed pt. A on it, let $AP=s$, $PQ=\delta s$, and the co-ordinates of P (x, y).



Mass of the element $PQ=\rho\,\delta s$.

$$\therefore\ \bar{x}=\frac{\int_l x\,.\,\rho\,ds}{\int_l \rho\,ds} \text{ and } \bar{y}=\frac{\int_l y.\rho ds}{\int_l \rho\,ds}$$

In case ρ is constant, *i. e.*, the wire is of uniform density.

$$\bar{x}=\frac{\int_l xds}{\int_l ds} \qquad \text{and} \qquad \bar{y}=\frac{\int_l yds}{\int_l ds}$$

Now in cartesian co-ordinates,

$$ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,.\,dx. \qquad \ldots\ldots(1)$$

In polar co-ordinates,

$$ds=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,.\,d\theta. \qquad \ldots\ldots(2)$$

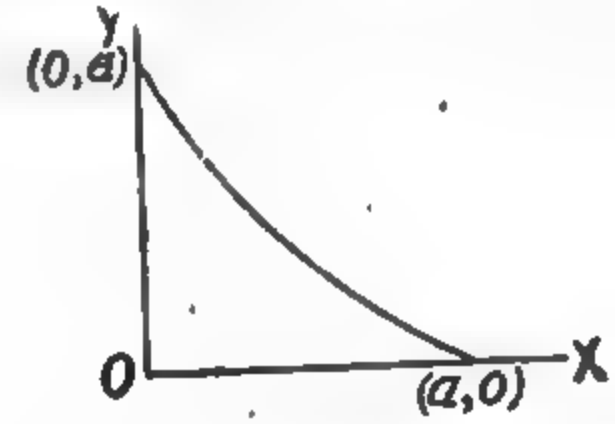
And for parametric equation of a curve in the form $x=f(t)$, $y=\phi(t)$

$$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt. \qquad \ldots\ldots(3)$$

Writing the expressions (1), (2), or (3) for ds according as the equation of the curve is cartesian, polar or parametric, we

can find the position of the C. G. of any given curve, the integration being performed within proper limits.

8.911. Example. (*i*) *Find the C. G. of the portion of the arc of the curve* $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, *lying in the first quadrant.*



Here $\frac{dy}{dx}=-\left(\frac{y}{x}\right)^{\frac{1}{3}}$

$$\therefore \quad ds=\sqrt{1+y^{\frac{2}{3}}/x^{\frac{2}{3}}}\,dx.$$

$$=\left(\frac{a}{x}\right)^{\frac{1}{3}}dx.$$

$$\bar{x}=\frac{\int_1 x\,ds}{\int_1 ds}=\frac{\int_0^a x\left(\frac{a}{x}\right)^{\frac{1}{3}}dx}{\int_0^a \left(\frac{a}{x}\right)^{\frac{1}{3}}ds}=\frac{\int_0^a x^{\frac{2}{3}}\,dx}{\int_0^a x^{-\frac{1}{3}}\,dx}=\frac{2a}{5}$$

By symmetry $\bar{y}=\bar{x}=\frac{2a}{5}$.

The same resalt can also be obtained by taking parametric equations $x=a\sin^3\theta$, $y=a\cos^3\theta$, of the same curve.

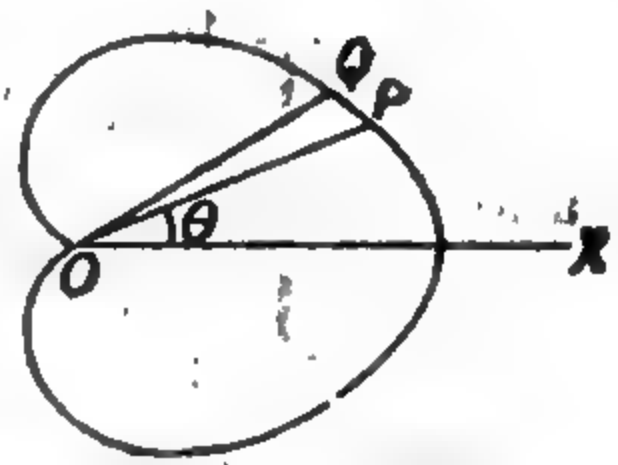
$$ds=\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}.\,d\theta=3a\sin\theta\cos\theta\,d\theta,$$

when $x=0$, $\theta=0$, and when $x=a$, $\theta=\frac{1}{2}\pi$.

$$\therefore\ \bar{x}=\frac{\int_1 x\,ds}{\int_1 ds}=\frac{\int_0^{\pi/2} a\sin^3\theta.3a\sin\theta\cos\theta\,d\theta}{\int_1 ds\quad\int_0^{\pi/2} 3a\sin\theta\cos\theta\,d\theta}=\frac{2a}{5}$$

$$\text{and}\ \ \bar{y}=\frac{\int_1 y\,ds}{\int ds}=\frac{\int_0^{\pi/2} a\cos^3\theta.\,3a\sin\theta\cos\theta\,d\theta}{\int ds\quad\int_0^{\pi/2} 3a\sin\theta\cos\theta\,d\theta}=\frac{2a}{5}$$

Example (ii) *Find the position of the C. G. of the arc of the cardiod* $r=(1+\cos\theta)$.



Here $ds=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}.\,d\theta$

$=\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\;d\theta$

$=2a\cos\frac{\theta}{2}\,d\theta$

$$\therefore \bar{x}=\frac{\int_1 x\,ds}{\int_1 ds}=\frac{\int_{-\pi}^{\pi} r\cos\theta.\,2a\cos\frac{\theta}{2}\,d\theta}{\int_{-\pi}^{\pi} 2a\cos\frac{\theta}{2}\,d\theta}$$

$$=2\int_0^{\pi} a(1+\cos\theta)\cos\theta\cos\frac{\theta}{2}d\theta\Big/2\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta.$$

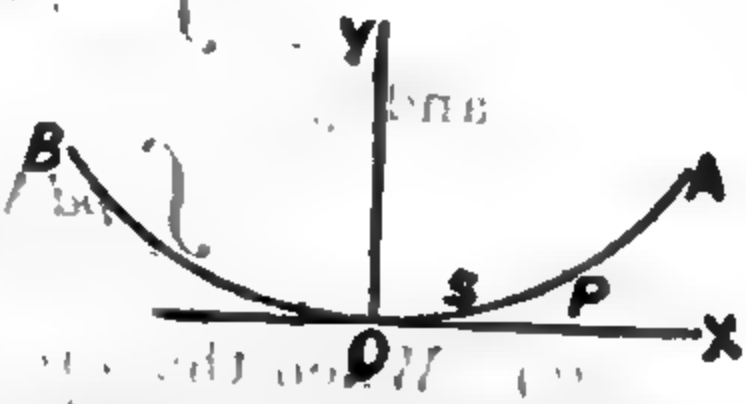
Putting $\theta=2\phi$ and simplifying, we have

$$=2a\int_0^{\pi/2}\cos^3\phi(2\cos^2\phi-1)d\phi\Big/\int_0^{\pi/2}\cos\phi\,d\phi=\frac{4a}{5}$$

By symmetry $\bar{y}=0$.

Example (iii) *If the density of a cyloidal arc* $s^2=8ay$, *varies as* s^2, *find the position of the C. G. of the cycloid.*

Here at any point P, the densiy $\rho=ks^2$



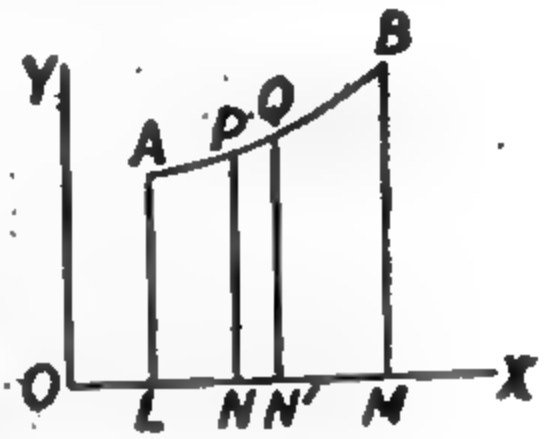
The whole length of the cycloid being $8a$, s varies from $-4a$ to $4a$.

By symmetry the C. G. lies on the y-axis, i.e., $\bar{x}=0$

and $$\bar{y}=\frac{\int_{-4a}^{4a} y(pds)}{\int_{-4a}^{4a} pds}=\frac{\int_0^{4a}\frac{s^2}{8a}\cdot ks^2ds}{2\int_0^{4a} ks^2ds}=\frac{1}{8a}\frac{\int_0^{4a} s^4ds}{\int_0^{4a} s^2ds}=\frac{6a}{5}$$

8.92. C. G. of a plane area.

(a) When the area is bounded by the curve $y=f(x)$, the axis of x and the ordinates $x=a$ and $x=b$.



We take the element of the area in the form of a strip PNN'Q of width δx parallel to the axis of y, at a distance x from it, δA, the area of the strip $=y\delta x$ and the co-ordinates of its C. G. are $\left(x, \frac{y}{2}\right)$, approximately.

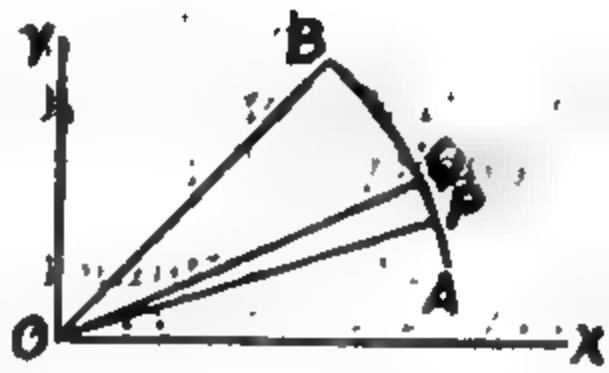
Let ρ be the mass per unit area, we have

$$\bar{x}=\frac{\int x\rho dA}{\int \rho dA}=\frac{\int_a^b \rho xy.dx}{\int_a^b \rho y.dx}$$

and $$\bar{y}=\frac{\int y\rho dA}{\int \rho dA}=\frac{\frac{1}{2}\int_a^b \rho y^2dx}{\int_a^b \rho ydx}$$

(b) When the area is bounded by the curve $r=f(\theta)$ and radii vectors $\theta=\alpha$ and $\theta=\beta$.

We take the element of the area in the form of the sector OPQ where OP$=r$ $\angle XOP=\theta$ and $\angle POQ=\delta\theta$

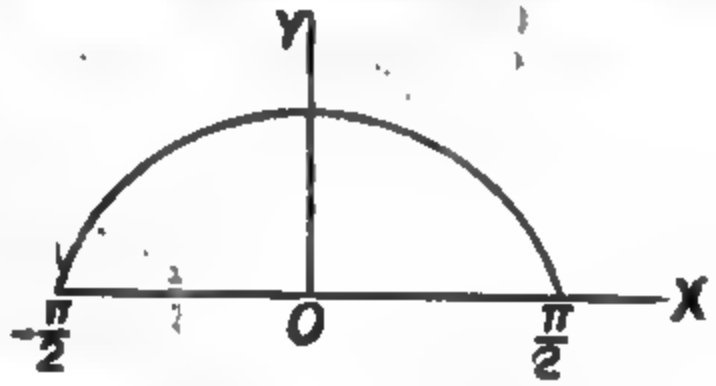


Here δA, area of the element $=\frac{1}{2}r^2\delta\theta$, and the co-ordinates of its C. G. are approximately $\frac{2}{3}r\cos\theta$, $\frac{2}{3}r\sin\theta$

Hence $$\bar{x}=\frac{\int_\alpha^\beta \frac{2}{3}r\cos\theta.\ \rho\frac{1}{2}r^2d\theta}{\int_\alpha^\beta \rho\frac{1}{2}r^2d\theta}=\frac{2}{3}.\frac{\int_\alpha^\beta \rho r^3\cos\theta\,d\theta}{\int_\alpha^\beta \rho r^2d\theta}.$$

and $$\bar{y}=\frac{\int_\alpha^\beta \frac{2}{3}r\sin\theta.\ \rho\frac{1}{2}r^2dx}{\int_\alpha^\beta \rho\frac{1}{2}r^2dx}=\frac{2}{3}.\frac{\int_\alpha^\beta \rho r^3\sin\theta d\theta}{\int_\alpha^\beta \rho r^2d\theta}$$

8.921. Example. (i) *Find the C. G. of a uniform lamina bounded by the curve $y=\cos x$, the axis of x, and the ordinates $x=-\pi/2$ and $x=\pi/2$.*



By symmetry the C. G. lies on the y-axis $\therefore\ \bar{x}=0$.

and $$\bar{y}=\frac{\frac{1}{2}\int ydx}{\int ydx}=\frac{\int_{-\pi/2}^{\pi/2}\cos^2x\,dx}{2\int_{-\pi/2}^{\pi/2}\cos x\,dx}=\frac{\pi}{8}.$$

Example. (ii) *Find the distance of the C. G. of the area of the cardiod $r=a(1+\cos\theta)$ from its cusp O, when the density varies as the nth power of the distance from O.*

(P.U. Hons. 1937.)

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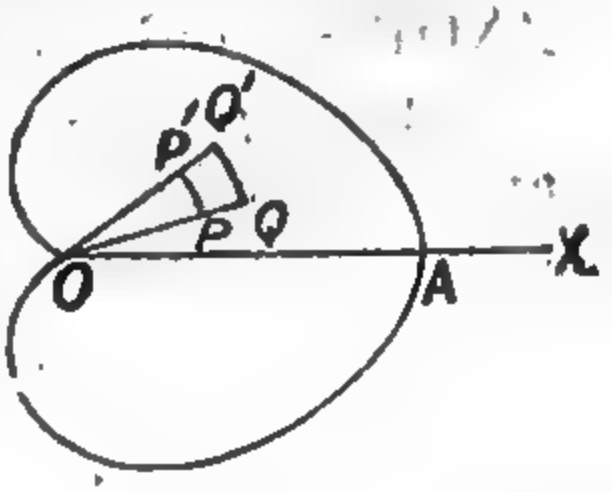
As the density v ries as r^n, we take the element of area in the form PQQ'P' of area $rd\theta dr$. The density ρ for the element $=kr^n$.



By symmetry the C. G. lies on OX, *i.e.*, $\bar{y}=0$ and $\bar{x}$ for the cardiod is the same as x of the C. G. of the upper half of the curve, for which θ varies from 0 to π and r from 0 to $r=a(1+\cos\theta)$.

$$\bar{x} = \frac{\int\int r\cos\theta.\ \rho r d\theta dr}{\int\int \rho r d\theta dr} = \frac{\int\int r^2\cos\theta.\ kr^n d\theta dr}{\int\int kr^n.r d\theta dr}$$

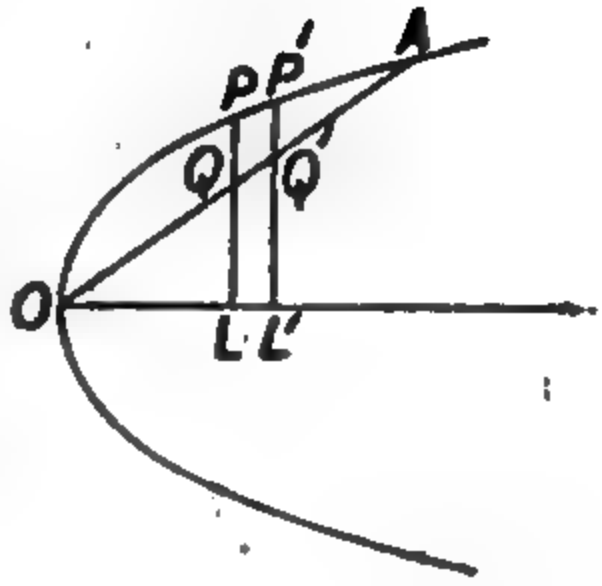
$$= \frac{\int_0^{\pi}\int_0^{r} r^{n+2}\cos\theta dr d\theta}{\int_0^{\pi}\int_0^{r} r^{n+1} dr d\theta} = \frac{(n+2)a\int_0^{\pi}(1+\cos\theta)^{n+3}\cos\theta d\theta}{(n+3)\int_0^{\pi}(1+\cos\theta)^{n+2}d\theta}$$

$$= \frac{2a(n+2)}{n+3}\cdot\frac{\int_0^{\pi/2}(\cos\phi)^{2n+6}(2\cos^2\phi-1)d\phi}{\int_0^{\pi/2}(\cos\phi)^{2n+4}d\phi} \quad \text{where } \theta=2\phi$$

$$= \frac{(n+2)(2n+5)a}{(n+3)(n+4)}.$$

In case the area is uniform $n=0$ and $\bar{x} = 5a/6$.

Example. (*iii*) *Find the C. G. of the area included between the parabola* $y^2=ax$ *and the line* $y=x$.



Co-ordinates of A, the point of intersection of the parabola and the straight line are (a, a).

We take an element PQQ'P' of the enclosed area. Let $LQ=y_1=x$ and $LP=y_2=\sqrt{ax}$.

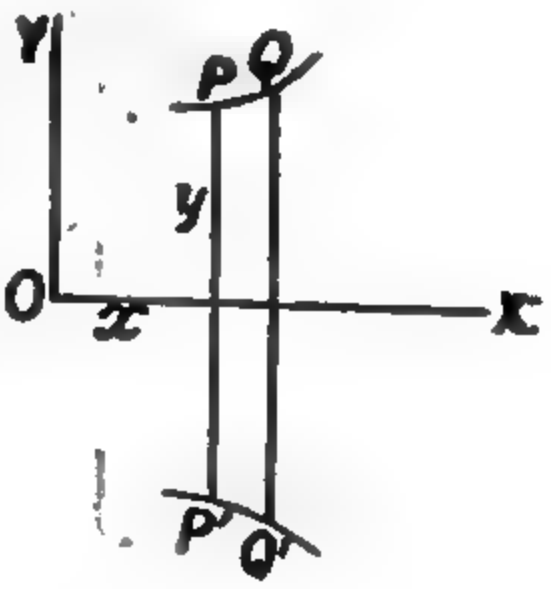
Area of the element $=(y_2-y_1)dx$ and dx being small, the co-ordinates of its C. G. are $\left(x, \frac{y_1+y_2}{2}\right)$.

$$\therefore \quad \bar{x}=\frac{\int_i x(y_2-y_1)dx}{\int(y_2-y_1)dx}=\frac{\int_o^a x(\sqrt{ax}-x)dx}{\int_o^a(\sqrt{ax}-x)dx}=\frac{2a}{5}$$

$$\text{and } \bar{y}=\frac{\int_i \frac{1}{2}(y_1+y_2)(y_2-y_1)dx}{\int(y_2-y_1)dx}=\frac{\frac{1}{2}\int_o^a(ax-x^2)dx}{\int_o^a(\sqrt{ax}-x)dx}=\frac{1}{2}a.$$

8.93. C. G. of the volume and surface of revolution.

Suppose a solid is formed by the revolution of the curve $y=f(x)$ about the axis of x, and suppose it is bounded by the ordinates $x=a$ and $x=b$.



(a) To find the *C. G. of the* **volume** *of the solid*, we take the element of its volume in the form of a disc PP'Q'Q, perpendicular to OX, at a distance x from O.

Since this disc is formed by the revolution of the ordinate $MP=y$ about OX,

volume of the disc of thickness $\delta x = \pi y^2 \delta x$, and by symmetry its C. G. lies on OX,

$$\therefore \quad \bar{y} = 0.$$

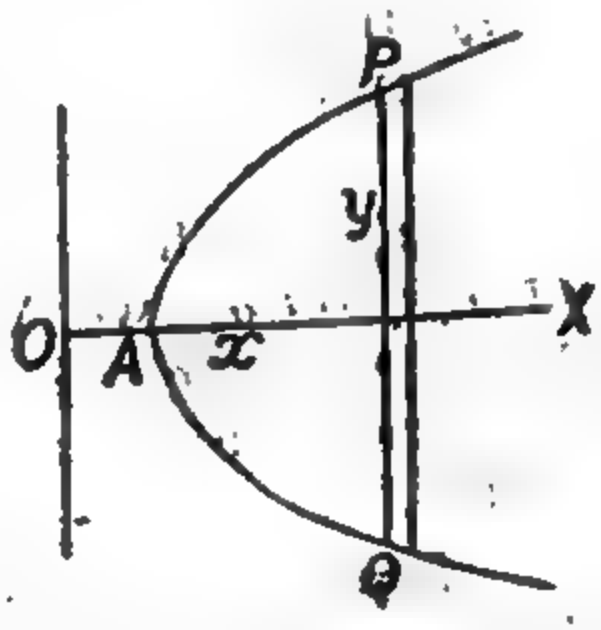
$$\text{and} \quad \bar{x} = \frac{\int x.\pi y^2 dx}{\int \pi y^2 dx} = \frac{\int_a^b x\{f(x)\}^2 dx}{\int_a^b \{f(x)\}^2 dx}$$

(*b*) To find the *C. G. of the* **surface** *of the solid*, we take the element in the form of a belt PP′Q′Q on the surface. The width of the belt $= l'Q = \delta S$ and its area $= 2\pi y \delta s$.

$$\therefore \quad \bar{x} = \frac{\int x.2\pi y ds}{\int 2\pi y ds} = \frac{\int xyds}{\int yds}$$

The limits for s being taken on the curve $y = f(x)$ corresponding to $x = a$ and $x = b$.

8.931. Example (*i*) *Find the C. G. of the volume formed by the revolution of the portion of the parabola* $y^2 = 4ax$, *cut off by the ordinate* $x = h$, *about the axis of* x. (*Agra*. 1936)

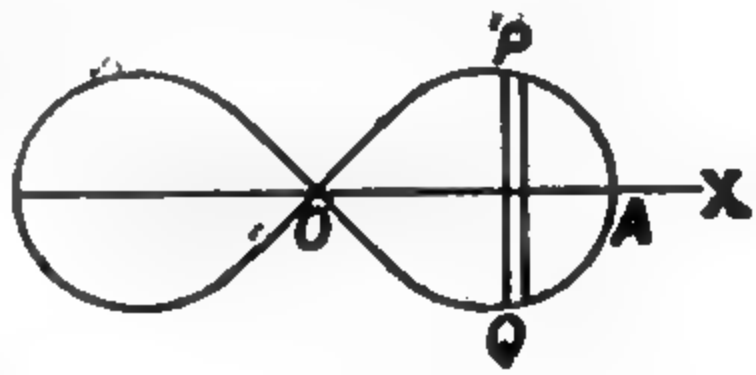


Here $$\bar{x} = \frac{\int x.\pi y^2 dx}{\int \pi y^2 dx} = \frac{\int_0^h x.4ax\,dx}{\int_0^h 4ax\,dx}$$

$$= \frac{\int_0^h x^2 dx}{\int_0^h x dx} = \tfrac{2}{3}h$$

Example. (iii) *Find the centroid of the surface formed by the revolution of one loop of the lemniscate* $r^2=a^2\cos 2\theta$, *about the initial line.*

The cartesian co ordinates (x, y) of any point $P(r, \theta)$ are given by $x=r\cos\theta$, $y=r\sin\theta$, and for the arc APO, θ varies from 0 to $\frac{\pi}{4}$.



$$\text{Also } \frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}=\sqrt{a^2\cos 2\theta+\frac{a^4\sin^2 2\theta}{a^2\cos 2\theta}}$$

$$=\frac{a}{\sqrt{\cos 2\theta}}.$$

$$\text{Now } \bar{x}=\frac{\int xyds}{\int yds}=\frac{\int_0^{\pi/4} r\cos\theta.r\sin\theta.\frac{a}{\sqrt{\cos 2\theta}}d\theta}{\int_0^{\pi/4} r\sin\theta.\frac{a}{\sqrt{\cos 2\theta}}d\theta}$$

$$=\tfrac{1}{4}a^3\int_0^{\pi/4} 2\sin 2\theta\sqrt{\cos 2\theta}\,d\theta\Big/a^2\int_0^{\pi/4}\sin\theta\,d\theta$$

$(\because r^2=a^2\cos 2\theta)$

$$=\frac{a}{4}\cdot\frac{\frac{2}{3}\Big|-\cos^{\frac{3}{2}}2\theta\Big|_0^{\pi/4}}{\Big|-\cos\theta\Big|_0^{\pi/4}}=\frac{a}{6\left(1-\frac{1}{\sqrt{2}}\right)}=\frac{a(2+\sqrt{2})}{6}$$

EXAMPLES XXXI

1. Find the position of the C. G. of the areas bounded by the following curves :—

(i) $ay^2=x^3$ between $x=0$ and $x=c$. [*Ans.* $x=5c/7$.

(*ii*) $y = \sin x$ between $x=0$, $x=\pi$ and the **X-axis.**

[*Ans.* $\bar{x}=\pi/2$, $\bar{y}=\pi/8$.

(*iii*) $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ between the **axes.**

[*Ans.* $\bar{x}=a/5$, $\bar{y}=b/5$.

(*iv*) $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$ in the positive quadrant.

Ans. $\frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{256}{315\pi}$

(*v*) $y^2(2a-x)=x^3$ and its asymptote. *Ans.* $\bar{x}=\frac{5a}{3}$.

(*vi*) the loop of $y^2(a+x)=x^2(a-x)$.

[*Ans.* $\bar{x}=a(3\pi-8)/3(4-\pi)$.

(*vii*) the cardiod $r=a(1+\cos\theta)$. *Ans.* $\bar{x}=5a/6$.

(*viii*) One loop of $r=a\cos 2\theta$. *Ans.* $\bar{x}=\frac{128\sqrt{2}a}{105\pi}$

(*ix*) One loop of $r^2=a^2\cos 2\theta$. *Agra. 1937.*

Ans. $\bar{x}=\frac{\pi a\sqrt{2}}{8}$.

2. Find the C. G. of a plate of uniform density forming a quadrant of an ellipse bounded by its semi-axes. (*Agra. 1939*)

[*Ans.* $(4a/3\pi, 4b/3\pi)$.

3. Find the C. G. of the area bounded by the parabola $y^2=4ax$, the axis of x and the latus-rectum. (*Ali. U.*)

[*Ans.* $3a/5$, $3a/4$.

4. If the density of a circular lamina varies as the nth power of the distance from a point O on the circumference, show that the C. G. of the lamina divides the diameter through O in the ratio $n+2 : 2$. (*P. U. Hons 1939*)

5. If the density of a complete circular arc varies as the square of the distance from a point O on the arc, prove that its C. G. divides the diameter through O in the ratio of 3 : 1.

6. Find the position of the C. G. of the (*i*) arc, (*ii*) area of the cycloid, $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$, which lies in the positive quadrant.

[*Ans.* (*i*) $\bar{x}=\left(\pi-\frac{4}{3}\right)a$. $\bar{y}=\frac{2a}{3}$. (*ii*) $\bar{x}=\frac{a}{2}\left(\pi-\frac{16}{9\pi},\right)$ $\bar{y}=\frac{7a}{6}$.

7. Find the C. G. of the areas enclosed by the curves :—

(*i*) $y^3=ax$ and $x^2=by$. [*Ans.* $9a^{\frac{1}{3}}b^{\frac{2}{3}}/20$, $9a^{\frac{2}{3}}b^{\frac{1}{3}}/20$.

(*ii*) $y^2=ax$ and $y^2=2ax-x^2$.

[*Ans.* $15\pi-44)\ a/5(3\pi-8)$, $a/(3\pi-8)$

(*iii*) $x^2+y^2-2ax=0$, and $x^2+y^2-2bx=0$, where a and b are positive. [*Ans.* $\frac{a^2+ab+b^2}{a+b}$, $\frac{4}{3}$. $\frac{a^2+ab+b^2}{\pi(a+b)}$

8. Find the C. G. of a solid formed by revolving the quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about its (*i*) major (*ii*) minor axis. (*Ans.* (*i*) $(\frac{3}{8}a, 0)$: (*ii*) $(0, \frac{3}{8}b)$.

9. Find the C. G. of the volumes formed by the revolution of :—

(*i*) $ay^2=x^3$, about the X-axis between the origin and $x=c$. [*Ans.* $\bar{x}=\frac{4}{5}c$.

(*ii*) cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$, about the axis of y. [*Ans.* $\bar{y}=\frac{63\pi^2-64}{9\pi^2-16}$. $\frac{a}{6}$.

(*iii*) $r=a(1+\cos\theta)$ about the X axis. [*Ans.* $\bar{x}=\frac{4a}{5}$.

10. Find the C. G. of the solid formed by the revolution of the area bounded by the porabola $y^2=4ax$, the X-axis and the latus rectum, about the latus rectum. (*All. U.* 1941).

[*Ans.* $\bar{x}=a$, $\bar{y}=5a/8$,

11. Find the C. G. of the surface formed by the revolution of $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ about the axis of y.

[*Ans.* $\bar{y}=\frac{2a}{15}\cdot\frac{15\pi-8}{3\pi-4}$.

12. Find the C. G. of a sphere of radius a whose density varies as the distance from a tangent plane.

[*Ans.* At a distance $6a/5$ from the plane.

13. Find the C. G. of a hemisphere of radius a whose density varies as the square of the distance from the centre of the complete sphere.

[*Ans.* At distance $5a/12$ from the base

14. Find the C. G. of the catenary $s=c\tan\psi$ between the points $\psi=0$ and $\psi=\pi/4$.

[*Ans.* $c[\log(1+\sqrt{2})-\sqrt{2}+1]$, $\frac{1}{2}c[\log(\sqrt{2}+1)+\sqrt{2}]$.

15. If G is the C. G. of any arc PQ of the Lemniscate $r^2=a^2\cos 2\theta$, prove that OG bisects the angle POQ, O being the pole of the curve.

16. Find the C. G. of a lamina in the form of a positive quadrant of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$. When the density at any point (x, y) is kxy. [*Ans.* $128a/429$, $128b/429$.]

17. Show that the C. G. of a circular disc, the density of which varies inversely as the fourth power of the distance from an external point O in its plane, is the inverse point of O with respect to the boundary of the disc.

18. Find the C. G. of a sold segment of a sphere, deduce the position of the C. G. of a body consisting of a cone and a segment of a sphere on the same base.

A solid sector of semi angle α is cut from a solid sphere and replaced with its edge in the same position as before but with its vertex outwards; show that the C. G. of the resulting system will coincide with the centre of the sphere if $\cos\alpha=3/5$. (*D. U. Hons.* 1938)

CHAPTER IX

VIRTUAL WORK. STABILITY OF EQUILIBRIUM

91. Work. When a body moves due to the action of a force the force is said to do work. In order to distinguish it from various other kinds of work, such work is called Mechanical work.

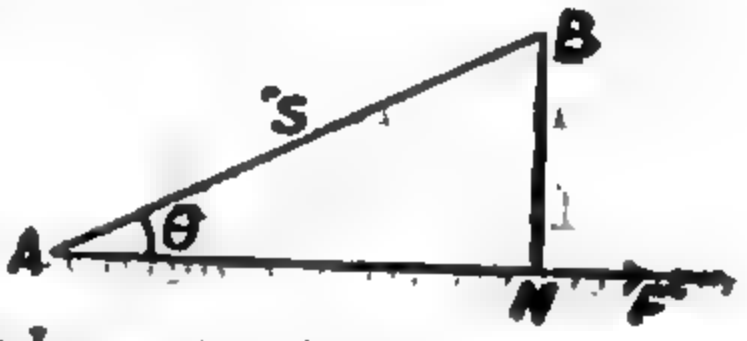
The work done by a constant force equal to the product of the force, and the displacement of its point of application in the direction of the force.

Let P be the force, s the displacement, and W, the work done.

If the displacement takes place in the direction and line of action of the force, then $W = F.s$.

If the displacement is in the same line as before, but in the opposite direction, the work done by the force is negative, i.e., $nW = -Fs$.

If the displacement takes place along a line inclined at an angle θ to the line of action of the force then



$$W = F.AN = F.s \cos\theta.$$

Since $F.s \cos\theta = s.F \cos\theta$,
the work done by the force is equal to the product of the displacement and the resolved part of the force in the direction of the displacement.

If θ is obtuse, W is negative. If $\theta = 90°$, $W = 0$, i.e a force does not work when the displacement takes place at right angles to its line of action.

As an illustration let W denote the work done by the force of gravity an a body of weight w.

When the body drops vertically a distance h, $W = wh$. When the body rises vertically a distance h, $W = -wh$. When

the body slides down a length l of an inclined plane of inclination a to the horizontal, $W = w.\ l \sin a = wh$, where h is the vertical distance in the direction of the force.

When $a=0$, $W=0$, *i.e.*, if the body moves along a horizontal plane the force of gravity does no work.

9.11. Work done by a variable force. Suppose the force varies and acts along the line AB on a particle which moves along that line. If F is the magnitude of the force at P at a distance x from A, the small amount of work δW, done by the force F when the particle moves a small distance δx from P, is approximately equal to $F\delta x$,

i.e.,
$$\delta W = F.\delta x,$$

$\therefore$ the total amount of work done by the variable force F when the particle moves from the point $x=a$ to the point $x=b$ is given by

$$W = \int_a^b F\, dx$$

9.12. Work done in stretching an elastic string.

Let one end of the string be fixed and let it be stretched by pulling it at the other end.

Let l be the natural length of the string and let it be stretched to a length $l+x$. The extension is x and the tension T of the string is $\lambda \frac{x}{l}$ which varies as x, the extension of the string beyond its natural length.

Now as the string is being stretched the force applied by an agent at any instant is always equal to the tension of the string at that instant.

$\therefore$ δW, the small amount of work done in stretching the string a further distance $\delta x = T\delta x$.

i.e.
$$\delta W = \lambda \frac{x}{l}\, \delta x.$$

$\therefore$ W, the total work done in stretching the string from a length l to a length $l+x$ is given by

$$W=\int_0^x \lambda \frac{x}{l}dx = \tfrac{1}{2}\frac{\lambda}{l}x^2 \qquad \text{.....(1)}$$

So that the work done in stretching an elastic string varies as the square of the extension.

Now the work done for an extension $x_1=\frac{1}{2}\frac{\lambda}{l}x_1^2$;

and the work done for an extension $x_2=\frac{1}{2}\frac{\lambda}{l}x_2^2$.

Hence the work done in increasing the extension from x_1 to x_2

$$=\tfrac{1}{2}\frac{\lambda}{l}x_2^2-\tfrac{1}{2}\frac{\lambda}{l}x_1^2$$

$$=\frac{\lambda}{l}\frac{x_1+x_2}{2}(x_2-x_1).$$

$\frac{\lambda}{l}x_1$ is the initial tension and $\frac{\lambda}{l}x_2$ is the final tension.

$\therefore \frac{\lambda}{l}\cdot\frac{x_1+x_2}{2}$ is the mean of the initial and final tensions and x_2-x_1 is the extension produced.

Hence *the work done in stretching an elastic string is equal o the product of the mean of the initial and final tensions and he extension.*

Since $\frac{\lambda}{l}\cdot\frac{x_1+x_2}{2}$ is also the tension for an extension $\frac{x_1+x_2}{2}$ therefore the work done is also equal to the product of the tension at $\frac{x_1+x_2}{2}$ and the extension.

9·2. *If a body of weight W undergoes a vertical displacement, such that its C. G. falls through a distance h, the work done is Wh.*

Let the body consist of an infinite number of particles of weights $w_1, w_2 \ldots\ldots w_n$. Let $z_1, z_2 \ldots\ldots z_n$ be the heights of the particles and z the height of the C. G. of the body above a given horizontal plane, in the initial position.

Let the same heights be $z_1', z_2', \ldots\ldots z'_n$ and $\bar{z}'$ in the second position.

Then $$W = w_1 + w_2 + \ldots\ldots w_n$$

$$W\bar{z} = w_1 z_1 + w_2 z_2 + \ldots\ldots + w_n z_n$$

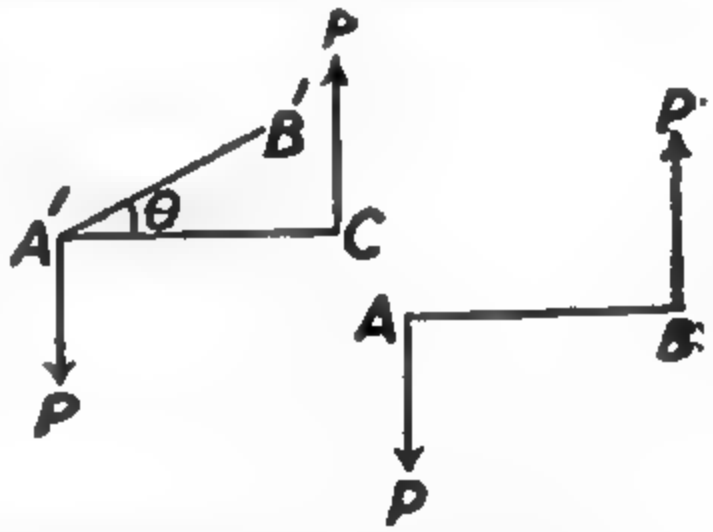
and $$W\bar{z}' = w_1 z_1' + w_2 z_2' + \ldots\ldots + w_n z_n'$$

$\therefore$ the work done by the weights of the particles

$$= w_1(z_1 - z_1') + w_2(z_2 - z_2') + \ldots\ldots + w_n(z_n - z_n')$$

$$= W(\bar{z} - \bar{z}') = Wh.$$

9·3. Work done by a couple. Let a couple of moments G be represented by two equal forces, each equal to P acting at the ends of the arm AB. Suppose that it is shifted to the position in which its arm takes up the position A′B′. Let θ be the angle between AB, A′B′, the two positions of the arm.



To arrive at the final position A′B′, suppose that the couple is at first displaced such that its arm AB is shifted to a parallel position A′C. The work done by the equal and opposite forces P during this displacement is zero.

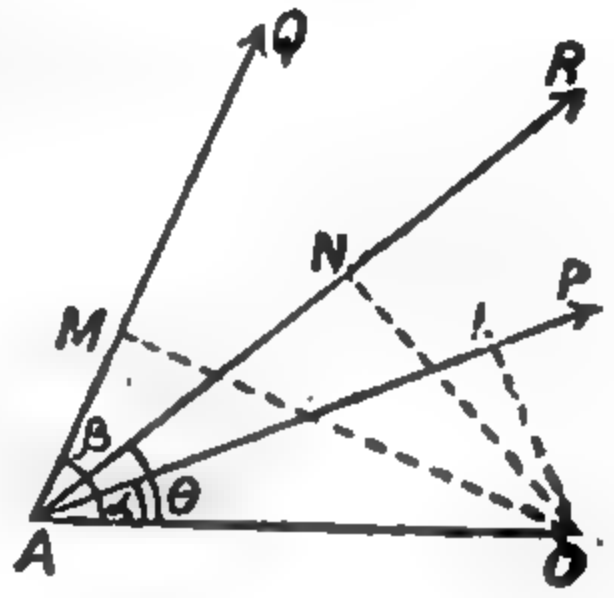
Now let the arm A′C turn about A′ through a small angle $\delta\theta$ so that the point C undergoes a small displacement equal to AC.$\delta\theta$ in the direction of the force P at C. The force P at A′ does no work as its point of application is not displaced and the small amount of work done by the other force is $P.A'C\delta\theta = P.AB\delta\theta = G.\delta\theta$.

Hence the total amount of work done by the couple till A′C has turned through a finite angle $= \int_0^\theta G.d\theta = G.\theta$.

i. e., the work done by a couple is equal to the product of the moment of the couple and the angle through which the arm of the couple turns.

9.4. *The sum of the works done by a number of co-planar forces acting on a particle in any displacement is equal to the work done by their resultant in the same displacement.*

Let the particle be displaced from A to O and let the forces P, Q.....etc., inclined at angle α, β,.....etc., with AO act on it. Let R be the resultant of the forces making an angle θ with AO.



The sum of the works done by the forces

$=P.\ AL+Q.\ AM+.....$

$=P.\ AO\cos\alpha+Q.\ AO\cos\beta+.....$

$=AO(P\cos\alpha+Q\cos\beta+.....)$

$=AO\times$sum of the resolved parts of the forces along AO.

$=AO\times$resolved part of the resultant along AO.

$=AO\times R\cos\theta=R.\ AO\cos\theta=$work done by the resultant.

Cor. 1. *If $R=0$ i. e., the forces are in equilibrium, the total work done by the forces vanishes.*

Cor. 2. *Since $P.AO\cos\alpha$*

$=P\times$projection of the displacement on the line of action of P,

it follows that if dp, dq.....etc., denote the projections of the displacements in the direction of the forces P, Q.....etc., and dr its projection in the direction of the resultant R, then

$$Rdr=Pdp+Qdq+.....=\Sigma\, pdp.$$

EXAMPLES XXXII

1. Show that the work done in dragging a body of weight W up a rough inclined plane of length l and inclination α is

$$lW(\sin\alpha+\mu\cos\alpha).$$

2. Prove that the work done in stretching an elastic string

of natural length l and modulus λ from tension T_1 to tension T_2 is

$$\frac{l}{2\lambda}(T_2^2 - T_1^2).$$

3. A spiral spring requires a force of one pound weight to stretch it one inch. How much work is done in stretching it 3 inches more ? [*Ans.* $\frac{5}{8}$ ft. lb.]

4. A body whose mass is m falls vertically to the earth's surface from a height a equal to the radius of the earth. Show that $\frac{1}{2}mga$ units of work is done by the earth's attraction which varies inversely as the square of the distance from the earth's centre.

5. A cylindrical cork of length l and radius a is slowly extracted from the neck of a bottle. If the normal pressure per unit area between the bottle and the unextracted part of the cork at any instant be constant and equal to P, show that the work done in extracting it is $\pi\mu al^2P$, where μ is the co-efficient of friction.

6. A load of two tons is suspended by a vertical chain, 300 ft. long, the chain itself weighing 6 lb. per ft. In winding up the load to the top, how many foot-pounds of work is done.

7. A block weighing 6 lb. rests on a rough horizontal plane, the co-efficient of friction being $\frac{1}{3}$. An elastic string of unstretched length 4 ft., whose length is doubled by a force equal to the weight of 8 lb. is fastened to the block and is pulled horizontally. How much work is done up to the instant when the block is just on the point of moving ?

Find also the total amount of work when the block has been dragged very slowly to a distance of 5 ft.

[*Ans.* 1 ft. lb. ; 11 ft. lb.]

9·5. Virtual Work. If a *particle* is in equilibrium under the action of a number of co-planar forces and the particle is displaced in any direction such that the forces remain constant in magnitude and direction then, the resultant being zero, the total work done by the forces is zero. (Art. 9·4.)

We now extend this principle to the case of a rigid body or a system of connected bodies. A number of problems in

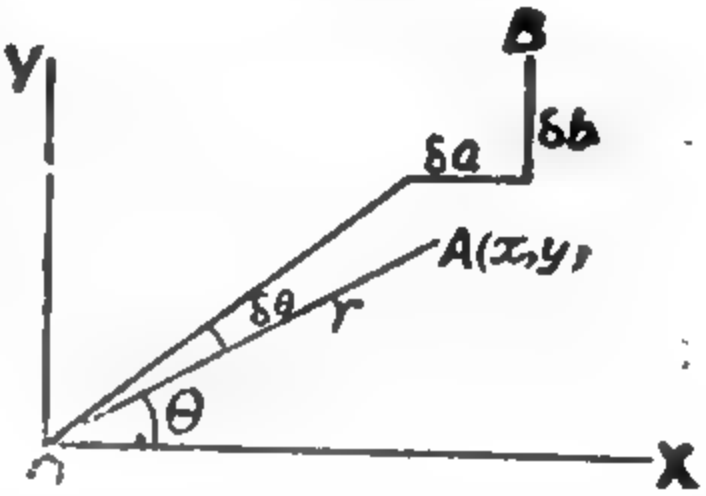
Statics are very much simplified by imagining the system at rest, to undergo a slight displacement and then equating the total work done by the forces to zero.

Since the displacement is only imaginary and not a real one, the total work so obtained is known as the *Virtual Work* and the method adopted is known as the *principle of Virtual Work*, which can be stated as follows :—

If a system of co-planar forces acting on a rigid body or a system of connected bodies is in equilibrium, and if the system undergoes a slight displacement consistent with its geometrical conditions, then the algebraic sum of the Virtual Works of the forces is zero and conversely if this algebraic sum is zero, the forces acting on the system are in equilibrium.

951. *Proof of the principle of Virtual Work for a system of co-planar forces acting on a rigid body.*

The most general displacement of a body is equivalent to a rotation about some point and a translation along some line.



Let the body be roated through an angle $\delta\theta$ about O and then translated distances δa, δb parallel to the fixed axes OX, OY where δa, δb and $\delta\theta$ are small quantities.

Let the point A with cartesian co-ordinates (x, y) and polar co-ordinates (r, θ), be the point of application of a force P before displacement and let B $(x+\delta x, y+\delta y)$ be the point of application of P after displacement,

then $x = r\cos\theta$ $\qquad y = r\sin\theta$

and $\delta x = r\cos(\theta+\delta\theta)+\delta a - r\cos\theta$

$= r(\cos\theta\cos\delta\theta - \sin\theta\sin\delta\theta) - r\cos\theta + \delta a$

$= -r\sin\theta.\delta\theta + \delta a$

$= \delta a - y\delta\theta$, neglecting small quantities of the second and higher orders.

Similarly $\qquad \delta y = r\sin(\theta+\delta\theta)+\delta b - r\sin\theta = \delta b + x\delta\theta.$

If X, Y be the components of P parallel to OX, OY, then the virtual work of P

$$=X\delta x+Y\delta y=X\ (\delta a-y\delta\theta)+Y\ (\delta b+x\delta\theta)$$
$$=X\delta a+Y\delta b+\delta\theta\ (xY-yX).$$

Similarly we obtain the virtual work of the other forces, δa, δb and $\delta\theta$ being the same for their displaced point of application.

$\therefore$ the Virtual Work of the whole system

$$=\delta a\Sigma X+\delta b\ \Sigma Y+\delta\theta\Sigma(xY-yX).$$

Now if the forces are in equilibrium then by Art. 6·1 $\Sigma X=0$, $\Sigma Y=0$ and $\Sigma(xY-yX)=0$ so that the virtual work vanishes.

Conversely, let the virtual work be zero

i. e., $\delta a\Sigma X+\delta b\Sigma Y+\delta\theta\Sigma(xY-y)=0$

for *all displacements*.

Considering, the displacement to take place parallel to X-axis *only*, so that $\delta b=0$ and $\delta\theta=0$,

then $\delta a\Sigma X=0$ or $\Sigma X=0$.

i.e., the sum of the resolved parts of the forces parallel to OX vanishes.

Similarly considering displacement parallel to Y-axis only we get $\Sigma Y=0$,

i.e, the sum of the resolved parts of the forces parallel to OY vanishes.

Finally considering displacement of pure rotation $\delta\theta$ about O, we get $\Sigma(xY-yX)=0$

i. e., the sum of the moments of the forces about O vanishes.

Thus the conditions of equilibrium are satisfied and therefore the system of forces is in equilibrium.

* **9·32.** Without neglecting second and higher powers of δa, δb and $\delta\theta$, we have

$$\delta x=r\cos(\theta+\delta\theta)+\delta a-r\cos\theta$$

$$= \delta a + x \cos \delta\theta - y \sin \delta\theta - x$$

$$= \delta a + x\left[1 - \frac{\delta\theta^2}{2\,!} + \frac{\delta\theta^4}{4\,!} \ldots\right] - y\left[\delta\theta - \frac{(\delta\theta)^3}{3\,!} + \ldots\right] - x.$$

$$= \delta a - y\delta\theta - \frac{x}{2\,!}(\delta\theta)^2 + \frac{y}{3\,!}(\delta\theta)^3 - \ldots\ldots$$

Similarly

$$\delta y = \delta b + x\delta\theta - \frac{y}{2\,!}(\delta\theta)^2 - \frac{x}{3\,!}(\delta\theta)^3 + \ldots\ldots\ldots$$

$\therefore$ work done by the force P

$$= X\delta x + Y\delta y$$

$$= X\delta a + Y\delta b + \delta\theta(xY - yX) - \frac{\delta\theta^2}{2\,!}(xX + yY) + \ldots\ldots$$

and the total work done

$$= \delta a \Sigma X + \delta b \Sigma Y + \delta\theta \Sigma(xY - yX) - \frac{\delta\theta^2}{2\,!}\Sigma(xX + Yy) + \ldots\ldots$$

We find that if the system is in equilibrium, *i. e.*, ΣX, ΣY, and $\Sigma(xY - yX)$ all vanish, the work done

$$= -\frac{(\delta\theta)^2}{2\,!}\Sigma(xX + yY) + \ldots\ldots$$

which is a small quantity of at least the second order as compared to $\delta\theta$.

Thus *for a system of forces in equilibrium, the virtual work is not exactly zero but is a small quantity of at least the second order*, conversely if the virtual work of a system of forces is a small quantity of at least the second order the forces are in equilibrium.

Example. For what sort of displacement is the work done by a system of co-planar forces in equilibrium, exactly zero ?

9·53. *Forces which may be omitted in the application of virtual work.*

The advantage of the principle of virtual work lies mainly in the fact that a certain number of forces can be avoided in writing the equations of equilibrium.

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The following are some of the important cases of forces which do not appear in the equation of virtual work.

1. *The reaction of any smooth surface upon a body which slides on the surface.*

The reaction being normal and therefore at right angles to the direction of displacement, will do no work

2. *The reaction of a fixed surface upon which a body rolls without sliding.*

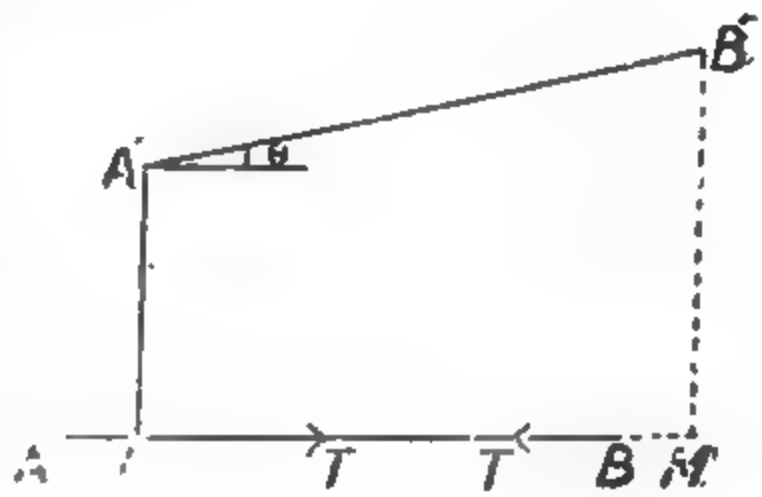
In this case, the point of contact remains instantaneously at rest, so that the displacement in the direction of the reaction is of an order higher than the first. Hence the work done by the reaction at the point of contact is negligible.

3. *The mutual reaction between any two bodies which roll upon one another.*

In this case both the bodies are taken into account and the action and reaction being equal and opposite, the virtual work is zero.

4. *The tension of an inextensible string.*

Let the string AB be displaced to the position A′B′, inclined at an angle θ to AB so that the points of application A and B of tension T are displaced to A′ and B′ respectively.



Draw A′L, B′M perpendicular to AB.

The work done

$$=T.AL-T.BM=T(AB-LB)-T(LM-LB)$$

$$=T.(AB-LM) =T.(AB-A'B' \cos\theta)$$

$$=T.AB(1-\cos\theta) \qquad \because A'B'=AB$$

$$=T.AB\left(1-1+\frac{\theta^2}{2!}+\ldots\ldots\right)$$

$=0$ to the first order of small quantity θ.

Note. In problems where the tension acting along an inextensible string is required, we suppose the string to be

removed and replaced by the force of tension T. The virtual work of T is then calculated consistent with geometrical conditions.

9 54. *If a body or a system of bodies are under the action of no force other than the weights of the bodies of the system and the mutual actions and reactions between hem, the positions of equilibrium correspond to the maxima or minima values of the height of the C. G. of the system above a fixed horizontal level*

Let W be the total weight, $\bar{z}$ the height of C G. of the system above a *fixed* horizontal level, $d\bar{z}$, the virtual displacement of the C. G.

Then, since the work of the mutual actions and reactions does not appear in the equation of virtual work, $\therefore$ by Art. 9·2 the virtual work of the system is $Wd\bar{z}$.

In an equilibrium position this vanishes,

$$\therefore \quad Wd\bar{z}=0 \qquad \text{or } d\bar{z}=0$$

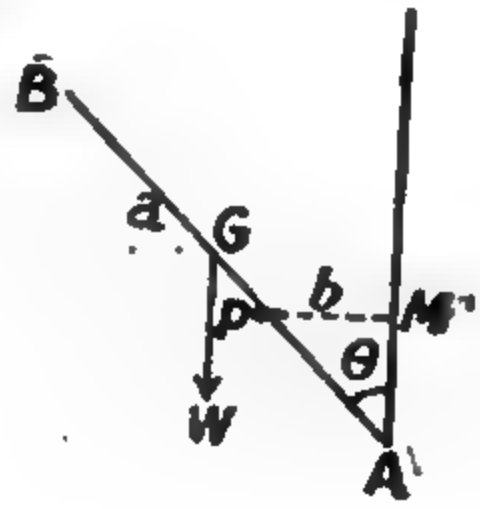
so that $\bar{z}$ is a maximum or a minimum.

It may be noted that since $W\bar{z}$ is the *potential energy* of the system, the *positions of equilibrium also correspond to the maxima or minima values of the potential energy of the system.*

9 55. The application of the principle of virtual work is illustrated by the following examples :—

Example. (i) *A uniform beam of length 2a, rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that in the position of equilibrium the beam is inclined to the wall at an angle* $\sin^{-1}\left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Let the rod AB(=2a) be inclined at an angel θ to the wall.



$\bar{z}$ the height of its C. G. above the fixed peg P=AG cos θ−AM=AG cos θ−PM cot θ

$$=a\cos\theta-b\cot\theta$$

$$\therefore\ d\bar{z}=(-a\sin\theta+b\,\mathrm{cosec}^2\theta)d\theta$$

For equilibrium $dz=0$,

$\therefore\ b\ \text{cosec}^3\theta - a\sin\theta = 0$

i.e., $\sin^3\theta = \frac{b}{a}$.

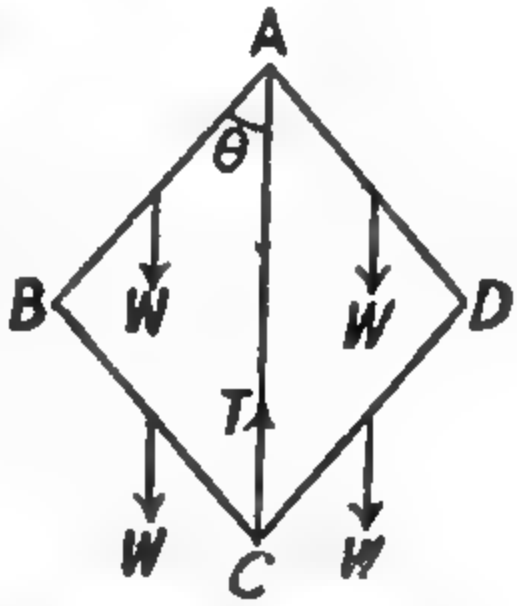
The same result can otherwise be obtained by taking moments about A and resolving vertically or merely from the geometry of the figure.

The simplicity of the method of virtual work lies in the fact that the reactions at A and P do not appear in the equation.

Example (*ii*) *Four equal heavy uniform rods are hinged together to form a rhoumbus ABCD and the frame is supported from A. The corners A and C are connected by an inelastic string to prevent the frame from collapsing. Find the tension in the string.*

Let $2a$ be the length, and W the weight of each rod.

A being fixed, we measure distances from it.



Let x be the depth below A of the C. G. of either of the upper rods. Then the depth of the C. G. of either of the lower rods is $3x$ and $AC=4x$, so that the virtual work of the weights of the two upper and two lower rods are

$$2W\delta x \quad \text{and} \quad 2W\delta(3x)=6W\delta x.$$

To find the tension in the string, we imagine the string to be removed and a force T applied at C along CA.

The virtual work of $T=-T\delta(4x)=-4T\delta x$.

The sign is taken negative, for the displacement of C is in a direction opposite to that in which T acts.

Hence the equation of virtual work is

$$2W\delta x+6W\delta x-4T\delta x=0 \quad i.e., \quad 8W\delta x=4T\delta x$$

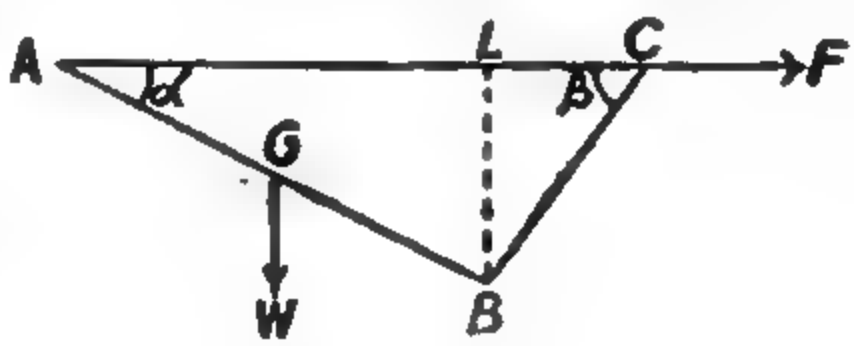
$$\therefore \quad T=2W.$$

Example. (*iii*) *A rod AB is movable about a pivot at A and to B is attached a string whose other end is tied to a ring. The ring slides along a smooth horizontal wire passing through A;*

prove that the horizontal force necessary to keep the ring at rest is

$$\frac{W \cos \alpha \cos \beta}{2 \sin (\alpha+\beta)},$$

where W is the weight of the rod and α, β the inclination of the rod and the string to the horizontal. (*P. U.* 1938, 48.)



Let a be the length of the rod and b that of the string, then

$$AC = a \cos \alpha + b \cos \beta,$$

and the depth of the C. G. of the rod below $A = \frac{a}{2} \sin \alpha$.

Here a and b remain constant and α and β vary.

$\therefore$ by the principle of Virtual work, we have

$$W\delta(\tfrac{1}{2}a \sin \alpha) + F\delta\,(a\cos \alpha + b \cos \beta) = 0$$

or $\frac{1}{2} Wa \cos \alpha\, d\alpha = F.\ a \sin \alpha\, d\alpha + F.b \sin \beta\, d\beta$ .....(1)

Also since $a \sin \alpha = BL = b \sin \beta$

$\therefore$ $a \cos \alpha\, d\alpha = b \cos \beta\, d\beta$ .....(2)

$\therefore$ $\frac{1}{2} Wa \cos \alpha = F\left(a \sin \alpha + \frac{a \cos \alpha \sin \beta}{\cos \beta}\right)$

i. e., $F = W \cos \alpha \cos \beta / 2 \sin (\alpha+\beta)$.

Note. The weight of the ring does not appear in the equation of virtual work as its point of application C slides along a horizontal line.

EXAMPLES XXXIII

1. A regular hexagon consists of six equal rods which are each of weight W and are freely jointed together. The hexagon rests in a vertical plane and AB is in contact with a horizontal table ; if C and F be connected by a light string, prove that its tension is $W\sqrt{3}$.

2. Four equal rods each of length a are hung from an angular point which is connected by an elastic string with the opposite point. If the rods hang in the form of a square and if the modulus of elasticity of the string be equal to the weight of a rod, find the unstretched length of the string.

Ans. $a\sqrt{2}/3$. (*D.U.* 1938)

3. A square frame work, formed of uniform heavy rods of equal weights W joined together, is hung by one corner. A weight W is suspended from each of the three lower corners and the shape of the square is preserved by a light rod along the horizontal diagonal. Find its tension.

[*Ans.* 4W (*All.U.*)]

4. AB and AC are uniform rods of weight w per unit length and their lengths are 6 and 8 units respectively. They are freely hinged at A and the ends B and C rest on a smooth horizontal table. Their middle points E and F are connected by a string 5 units. Find the tension of the string. (*Ali. U.* 1943)]

[*Ans.* $6\frac{18}{25}w$.

5. ABCD is a parallelogram of freely jointed rods, a point P on AB is joined to a point Q on CD by a string and R on AD to S on BC by another string. If the tensions in the string be T and T′, show that for equilibrium

$$\frac{T}{PQ}\cdot\frac{AP-DQ}{AB}=\frac{T'}{RS}\cdot\frac{BS-AR}{AD}.$$

(*D. U.* 1925)

6. A frame, formed of four light rods, each of length a freely jointed at A, B, C, D is suspended at A. A mass m is suspended from B and D by two strings each of length l and $l > \frac{a}{\sqrt{2}}$. The frame is kept in the form of a square by a string AC. Apply the method of virtual workto find the tension T in AC and show that when $l=a\sqrt{5}$, $T=^2mg$.

(*I.C.S.* 1928)

$$\left[Ans.\ \frac{m}{2}\left(1+\frac{a}{\sqrt{2l^2-a^2}}\right)\right.$$

7. A frame work of n equal *weightless* rods freely jointed together is maintained in the form of a regular polygon by equal forces P applied at the middle point of each rod at right angles to it. Prove that the tension or thrust in each rod is $\frac{1}{2}P\cot\frac{\pi}{n}$.

(*Bombay* 1934).

8. ABCD is a rhombus formed with four rods each of length l and weight w, joined by smooth hinges. A weight

W is attached to the lowest hinge C and the frame rests on two smooth pegs in a horizontal line and B and D are joined by a string. If the distance of the pegs apart i $2d$, and the angle at A is 2α, show that the tension in the string is

$$\tan\alpha\left[\frac{d}{2l}(W+4w)\operatorname{cosec}^3\alpha-(W+2w).\right] \quad (Agra\ 1936)$$

9. Two equal rods AB, BC, each of weight W, are freely jointed at B and stand in a vertical plane with their feet A, C resting on a smooth horizontal plane. They are kept in position by a *light* rod connecting A with the middle point of BC. By the method of virtual work show that the tension in the connecting rod is $\frac{1}{4}W\sqrt{1+9\tan^2\theta}$.

10. A smooth sphere of radius r and weight W rests in a horizontal circular hole of radius a. A string is wrapped once round the sphere above the hole and pulled tight. Show that the tension in the string which will just raise the sphere is

$$\frac{W}{2\pi}\cdot\frac{a}{\sqrt{r^2-a^2}}. \quad (Prescott)$$

11. A solid hemisphere is supported by a string fixed to a point on its rim and to point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If ϕ, θ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove by the application of the principle of virtual work that

$$\tan\phi=\tfrac{3}{5}+\tan\theta. \quad (P.\ U.\ Hons.\ 1937).$$

12. A flat semi-circular board with its plane vertical and curved edge upwards rests on a smooth horizontal plane and is pressed at two given points of its circumference by two beams which slide in smooth vertical tubes. If the board is in equilibrium, find the ratio of the weights of the beams.
(*P. U. Hons.* 1934)

[*Ans.* $\tan\theta : \tan\phi$ where θ, ϕ are the angles made with the horizontal by the radii through the points of contact.

13. A smooth circular cylinder of radius b is fixed parallel to a smooth wall with its axis horizontal and at a distance c

from the wall. A smooth uniform heavy rod of length $2a$ rests on the cylinder with one end on the wall and in a plane perpendicular to the wall show that its inclination θ to the horizontal is given by $a \cos^3 \theta + b \sin \theta = c$.

14. Two small smooth rings of equal weight slide on a fixed elliptic wire, whose major axis is vertical, and they are connected by a string which passes over a small smooth peg at the upper focus. Show that the rings will be in equilibrium wherever they are placed. (*Agra* 1938)

15. An endless chain of weight w rests in the form of a circular band round a smooth vertical cone which has its vertex upwards. Find the tension in the chain due to its own weight, assuming the vertical angle of the cone to be $2a$.

[*Ans* $\frac{1}{2\pi} w \cot a$.

16. A freely jointed frame work is formed of five equal uniform rods each of weight W. The frame work is suspended from one corner which is also joined to the middle point of the opposite side by an inextensible string., if the two upper and the two lower rods make angles θ and ϕ respectively, with the vertical, prove that the tension of the string is to the weight of the rod as

$$4 \tan \theta + 2 \tan \phi : \tan \theta + \tan \phi.$$ (*D.U.* 1924)

17. A string of length a forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods be supported in a horizontal position prove that the tension of the string is $\frac{2W}{b} \cdot \frac{2b^2 - a^2}{\sqrt{(b^2 - a^2)}}$. (*D. U.* 1933).

18. Three equal uniform rods AB, BC, CD each of weight w are freely jointed together at B and C and rest in a vertical plane, A and D being in contact with a smooth horizontal table. Two equal light strings AC and BD help to support the framework so that AB and CD are each inclined at an angle a to the horizontal. Show that if a mass of weight W be placed on BC at its middle point, then the tension of each string will be of magnitude

$$(w + \tfrac{1}{2}W) \cos a \operatorname{cosec} \tfrac{1}{2}a.$$

19. Three rigid rods AB, BC, CD each of length $2a$, are smoothly jointed at B and C. The system is placed in a vertical plane so that the rods AB CD are in contact with two smooth pegs distant $2c$ apart in the same horizontal line, the rods AB, CDmake equal angles α with the horizon. Prove that the tension of the string AD which will mainta n this configuration is.

$$\tfrac{1}{4}W \operatorname{cosec} \alpha \sec^2 \alpha \left\{\frac{3c}{a} - (3+2\cos^3 \alpha)\right\} \quad (D.U.\ 1926).$$

20. AB, BC are two beams, of weight W_1 and W_2 respectively, freely jointed at B and standing in a vertical plane with their ends A and C on a smooth horizontal plane, equilibrium being maintained by a light rope joining their midpoints A load W_3 is placed at B, and the centres of mass of the beams are at D and E, respectively, where AD : DB = BE : EC = $a : b$. If θ is the angle BAC and ϕ the angle BCA, show that the tension in the rope is

$$2[aW_1 + bW_2 + (a+b)\ W_3]/(a+b)\ (\tan\theta + \tan\phi).$$

21. Two equal rods each of weight wl and length l are hinged together and placed astride a smooth horizontal cylindrical peg of radius r. Then the lower ends are tied together by a string and the rods are kept at the same inclination ϕ to the horizontal. Find the tension in the string, and if the string is slack, show that ϕ satisfies the equation

$$\tan^3\phi + \tan\phi = l/2r.$$

22. Two smooth wires OA and OB are fixed in a vertical plane and are inclined at equal angles α to the vertical. Two equal heavy uniform rods HC and KC are smoothly hinged at C and the ends H and K are constrained to slide along OA and OB respectively by means of small smooth weightless rings; the weight of each rod is W, and an additional weight W is attached at C. Show that in the symmetrical position of equilibrium each rod is inclined at an angle θ to the vertical given by $\tan\theta = \frac{3}{2}\cot\alpha$. (*I. S. E.*)

23. A roof truss of two equal beams AB, AC tied together by light rods joining B, C to E, D, points in AC, AB respectively, such that AD = AE = AB/n. The weight of the roof and the truss together may be assumed to be uniformly

distributed along AB, AC and to be of amount w per unit length. Prove that the stress in CD or BE is

$$\tfrac{1}{3}\, n\, w \sec a.\ \mathrm{CD}$$

where a is the angle of inclination of AB or AC to the vertical. It is supposed that there is no horizontal reaction between the wall and the truss.

9.6. Principle of Energy. *When a particle undergoes displacement under the action of a conservative system of coplanar forces, the sum of its kinetic and potential energies remains constant.*

At any point P (x, y) occupied by the particle at any instant let ds be the displacement and F be the component of the forces in the direction of ds, then the small work done by the forces is $F.ds$.

If X, Y be the component of the forces in the direction of the co-ordinate axes and dx, dy the displacements in these directions, then by Art. 9.4 the work done is also equal to $X dx + Y dy$.

∴ the total work done in a finite-displacement when the particle moves from a point A (x_1, y_1) to another point B (x_2, y_2) is

$$\int_A^B F ds = \int_A^B (X dx + Y dy) \qquad \ldots\ldots(1)$$

If v be the velocity of the particle in the position P and m be its mass, then $F = mv \dfrac{dv}{ds}$.

$$\int_A^B F ds = \int_A^B mv \frac{dv}{ds}\, ds = \Big|\ \tfrac{1}{2}mv^2 \ \Big|_{u_1}^{u_2}$$

$$= \tfrac{1}{2}mu_2^2 - \tfrac{1}{2}mu_1^2$$

$$= K_B - K_A \ \ldots\ldots (2)$$

where u_1, u_2 are the velocities of the particle at A and B respectively.

The expression $\frac{1}{2}mv^2$ is defined to be the **Kinetic Energy** of a body of mass m, moving with a velocity v.

Thus the left hand side of equation (1) *i.e. the work done by the f rces, is equivalent to the change in the kinetic energies* in the two po: itions. We denote it by $K_B - K_A$ where K stands for the kinetic energy.

Now imagine that there is some function of x and y denoted by V such that, $X = -\frac{\partial V}{\partial x}$ and $Y = -\frac{\partial V}{\partial y}$.

$$\int_A^B (Xdx + Ydy) = -\int_A^B \left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy\right)$$

$$= -\int_A^B dV = V_A - V_B, \quad \ldots\ldots(3)$$

So that the nature of the forces is such that the work done by them depends only on the initial and final positions A and B and does not depend on the path followed by the particle. Such forces are known as *Conservative Forces and V is known as the* **Potential Function.**

Thus the right hand side of equation (1) *i.e., the work done by forces is equivalent to the change in the Potential Energy of the particle.*

From (2) and (3), we have

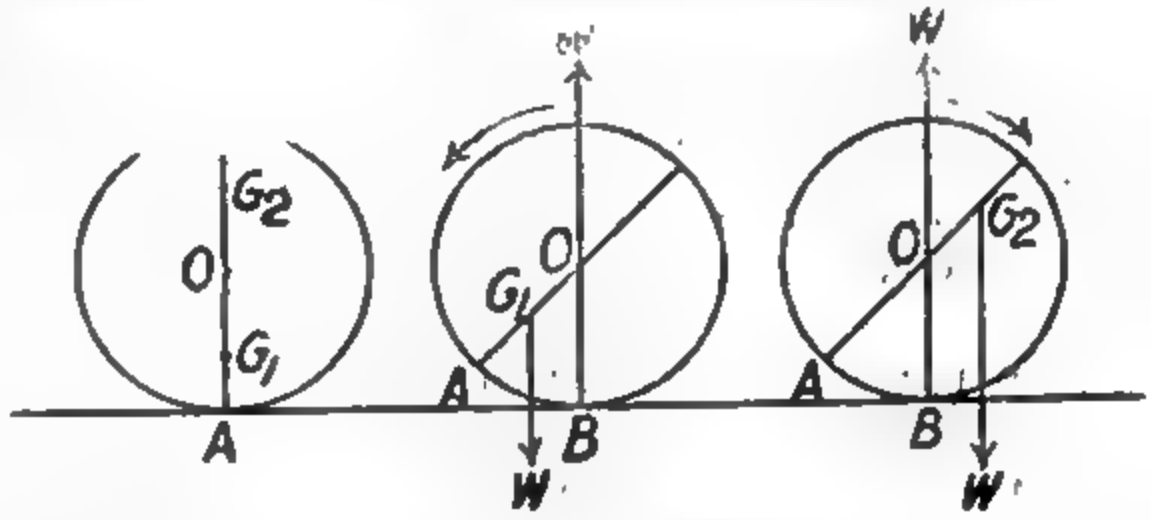
$$K_B - K_A = V_A - V_B \text{ or } K_B + V_B = K_A + V_A$$

i.e., the sum of the kinetic and potential energies remain constant.

This theorem can easily be extended for a system of particles.

9·7. Stability of Equilibrium. Let us consider the case of a sphere resting on a horizontal table. If it is not uniform its C. G. will be at a point different from O, the centre of the sphere.

Suppose it is placed such that its C. G. is vertically above or below O. The weight being balanced by the equal and opposite reaction at the point of contact A, the sphere is in equilibrium. Suppose now it is slightly displaced from this position, two cases arise according as its C. G. is above or below O.



(*i*) When the C. G. is below O, say at G_1, the weight W acting at G_1 and the equal and opposite reaction at B form a couple which tends to bring the sphere back to its original position of equilibrium. In this case the equilibrium of the sphere is said to be **stable.**

(*ii*) When the C. G. is above O, say at G_2, the weight W acting at G_2 and the equal and opposite reaction at B form a couple which tends to displace the sphere away from its original position of equilibrium. In this case the equilibrium is said to be **Unstable.**

Now suppose the sphere to be uniform, so that its C. G. is at the centre O. The reaction and the weight both pass through the centre and act along the same vertical line, so that even if the sphere is displaced the condition of equilibrium still holds and the sphere continues to remain in equilibrium. In such cases the equilibrium is said to be **Neutral.**

9·71. Conditions of Stability of Equilibrium. We suppose that the position of a rigid body or a system of connected bodies depends upon some variable θ, then V the potential energy of the system is a function of θ.

Now suppose that for a certain value of θ, the system is in equilibrium under the action of a system of coplanar forces. Let θ change to $\theta+\delta\theta$, so that the system undergoes a slight displacement consistent with its geometrical conditions.

Let δV be the change in the P. E. which is also the magnitude of the amount of work done in displacing the system. By the principle of virtual work, we have $\delta V=0$ *i.e.*, V is a maximum or minimum for equilibrium.

When released from its displaced position the system, being not in epuilibrium, must move and acquire kinetic energy.

By Art 9.6 this kinetic energy is acquired at the expense of the potential energy, so that the potential energy of the system will decrease.

Now we consider two cases :—

(1) Let the P. E. of the system in its original position of equilibrium be a minimum.

In this case as the system moves from its displaced position, the P. E decreases, and tends to attain its original minimum value so that the system returns to its former position and therefore, the original position of equilibrium was *stable.*

(2) Let the P. E. of the system in its original position be a maximum. In this case as the system moves from its displaced position, the P. E. decreases, so that the system tends to move away from its original position of maximum P. E, and therefore the equilibrium was *unstable.*

Thus we have proved that *the positions of equilibrium of a system are the positions of its maximum or minimum potential energy, the equilibrium being stable or unstable according as the potential energy is a minimum or a maximum.*

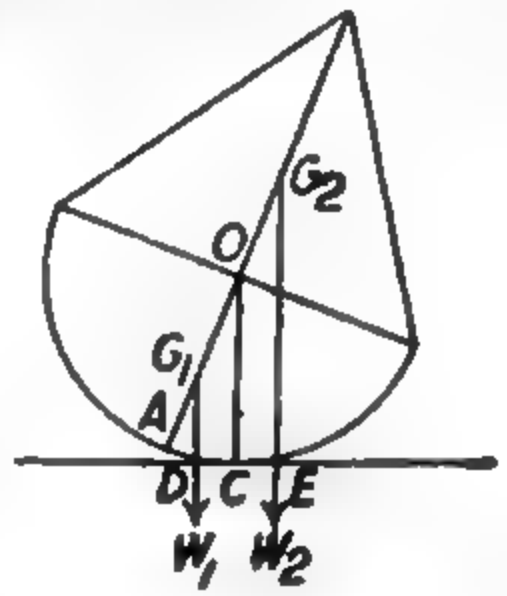
9·72. In Art. 9·54, we proved that in problems where we are concerned mainly with the weights of the system, the system is in equilibrium when $\bar{z}$ the height of its C. G. above a fixed horizontal level is a maximum or a minimum *i.e.*, $\frac{d\bar{z}}{d\theta}=0$. We now learn that the equilibrium will be stable or unstable according as $\bar{z}$ is a minimum or a maximum, *i.e.* $\frac{d^2\bar{z}}{d\theta^2}$ is positive or negative.

In certain simple cases such as given in Art. 9·7 the stability of equilibrium can be easily determined by taking moments about the point of contact.

9.73. Example (*i*) *A homogeneous solid is built up of a hemisphere of radius a and right circular cone of radius a and height h on the same circular base as the hemisphere. The solid*

is placed upright on a horizontal plane. When is the equilibrium stable ?

Let G_1, G_2 be the centres of gravity of the hemisphere and the cone, C the point of contact of the body in the displaced position.



The surface of contact being spherical, the reaction must pass through O, the centre of the sphere.

Now $OG_1 = \frac{3}{8}a$, $OG_2 = \frac{1}{4}h$.

The equilibrium will evidently be stable if $W_1 DC > W_2 . CE$.

The solid being uniform W_1, W_2 are proportional to the volumes of the hemisphere and the cone. Also DC and CE are proportional to OG_1 and OG_2.

$\therefore$ for stability

$$\tfrac{2}{3}\pi a^3 \times \tfrac{3}{8}a > \tfrac{1}{3}\pi a^2 h \times \tfrac{1}{4}h$$

or $$a^2 > \tfrac{1}{3}h^2 \qquad i.e.\ h < \sqrt{3}\, a.$$

Example (*ii*) *In Ex.* (*i*) *Art 9.55, prove that the equilibrium is unstable.*

It has been shown that

$$\bar{z} = a\cos\theta - b\cot\theta,$$

$\frac{d\bar{z}}{d\theta} = -a\sin\theta + b\,\mathrm{cosec}^2\theta$ and $\sin^3\theta = \frac{b}{a}$ for equilibrium.

Differentiating again, we have

$$\frac{d^2\bar{z}}{d\theta^2} = -a\cos\theta - 2b\,\mathrm{cosec}^2\theta\cot\theta.$$

$$= -\cos\theta\,(a + 2b\,\mathrm{cosec}^3\theta) = -\cos\theta\left(a + 2b \times \frac{a}{b}\right)$$

$= -3a\cos\theta$, which is negative, θ being acute.

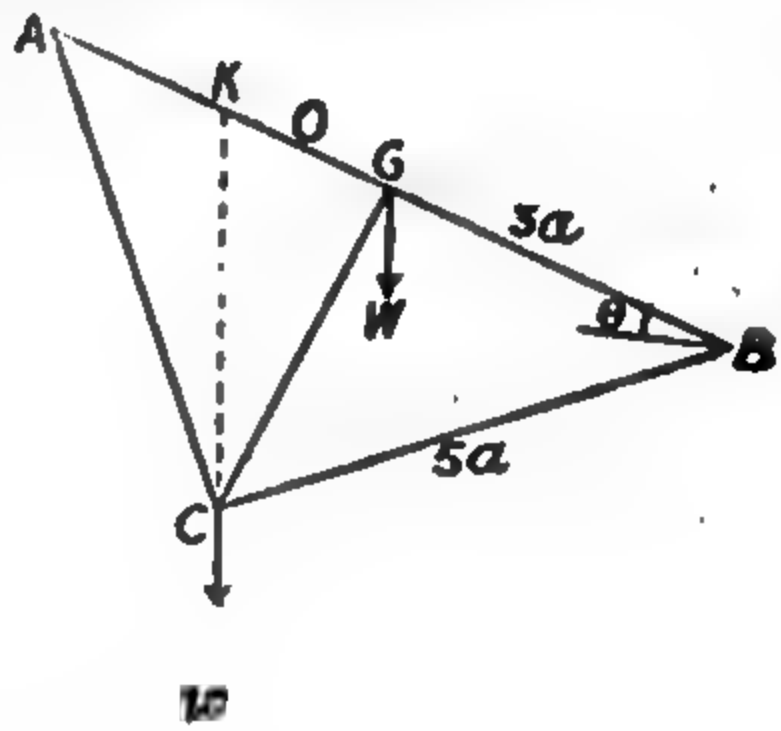
Thus $\bar{z}$ is a maximum, and equilibrium is unstable.

Example (*iii*) *AB is a uniform rod of length 6a and weight W, which can turn freely about a fixed point in its length distant 2a from A. AC and BC are light strings of length 5a attachde*

to a particle C of weight w. Show that if W is less than 2w there will be stable equilibrium with AB inclined to the horizontal at an angle

$$\tan^{-1}\frac{W+w}{4w}.$$

Let AB be the rod, O the point about which it turns, G its C. G and θ its inclination to the horizon.



Since $BG=3a$, $CB=CA=5a$
$\therefore$ $CG=4a$.

Also since $AO=2a$, $AG=3a$
$\therefore$ $OG=a$.

The depth of G below the horizontal level through the fixed point O.
$=OG \sin\theta = a\sin\theta$;
and the depth of C below O
$=OG\sin\theta+CG\cos\theta$
$=a\sin\theta+4a\cos\theta$.

$\therefore$ V, the potential energy of the system
$=-Wa\sin\theta-w(a\sin\theta+4a\cos\theta)$

$$\therefore \frac{dV}{d\theta}=-Wa\cos\theta-wa\cos\theta+4aw\sin\theta$$

Equating this to zero, we obtain $\tan\theta=\frac{W+w}{4w}$, which gives the position of equilibrium.

$$\frac{d^2V}{d\theta^2}=(W+w)a\sin\theta+4aw\cos\theta$$

which is positive, so that V is minimum and equilibrium is stable,

Now let the vertical through C meet AO in K, for equilibrium K must lie between O and A *i.e.*, $OK<2a$.

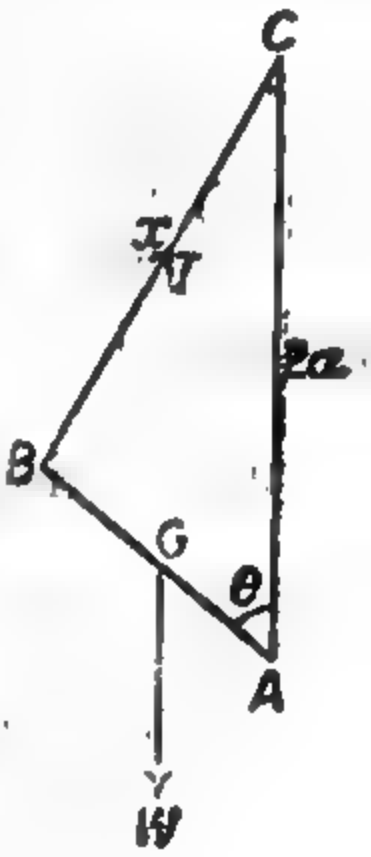
Taking moments about O, we have $w.OK=WOG=aW$

$$\therefore OK=\frac{aW}{w}$$

But since $OK<2a$, $\therefore$ $W<2w$.

Example (iv) *A uniform rod AB of weight W and length 4a is smoothly hinged at the end A, while B is connected to a fixed point C, which is at a distance 2a vertically above A, by a light elastic string of nature length ½ a. If the modulus of elasticity of the string is ⅕ W, prove that there are three positions of equilibrium in one of which the rod is inclined to the vertical at an angle* $\cos^{-1}(29/36)$. *Examine their stability.*

Let x be the length of the string BC, and θ, the angle whlch the rod AB makes with the vertical in the position of equilibrium.



The height of the C. G. of the rod above $A = \frac{a}{2}\cos\theta$.

The Pot. energy of the elastic string

$$= \tfrac{1}{2}\,\frac{W}{5}\,\frac{(x-\frac{1}{2}a)^2}{\frac{1}{2}a}. \qquad \text{(Art. 9.12)}$$

$\therefore$ V, the Pot. energy of the system

$$= W.\frac{a}{2}\cos\theta + \frac{W}{5a}(x-\tfrac{1}{2}a)^2$$

$$\therefore \quad \frac{dV}{d\theta} = -\tfrac{1}{2}Wa\sin\theta + \frac{2W}{5a}\left(x-\frac{a}{2}\right)\frac{dx}{d\theta}$$

But from the figure, $x^2 = 4a^2 + a^2 - 4a^2\cos\theta$, ...(1)

$$\therefore \quad x\frac{dx}{d\theta} = 2a^2\sin\theta. \qquad \text{.....(2)}$$

Hence $$\frac{dV}{d\theta} = -\tfrac{1}{2}Wa\sin\theta + \frac{2W}{5a}.2a^2\sin\theta - \frac{W}{5}\frac{2a^2\sin\theta}{x}.$$

$$= \frac{2}{5}aW\sin\theta\left(\frac{3}{4}-\frac{a}{x}\right) \qquad \text{.....(3)}$$

Also $$\frac{d^2V}{d\theta^2} = \frac{2}{5}aW\cos\theta\left(\frac{3}{4}-\frac{a}{x}\right) + \tfrac{2}{5}aW\sin\theta.\frac{a}{x^2}.\frac{2a^2\sin\theta}{x} \qquad \text{...(4)}$$

Equating (3) to zero, we get $\sin\theta=0$, *i.e.*, $\theta=0$, or π and $x=4a/3$.

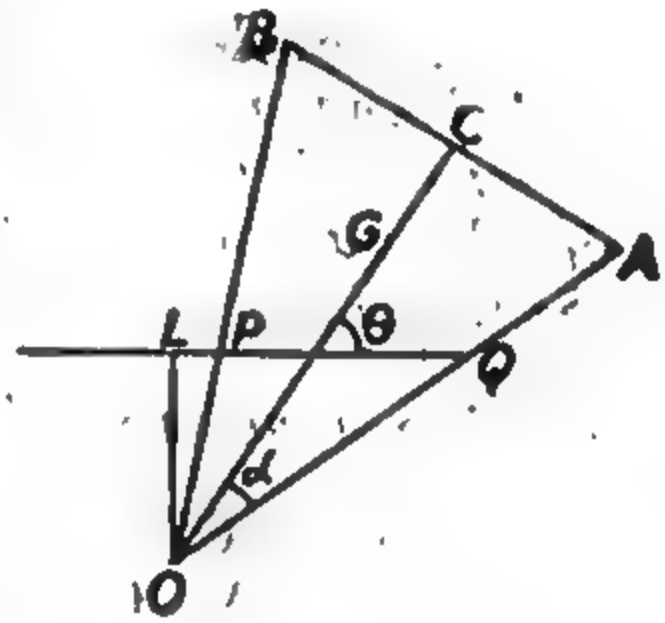
$\therefore$ when $\theta=0$, $x=a$, $d^2V/d\theta^2$ is—ve, equilibrium is unstable; when $\theta=\pi$, $x=3a$, $d^2V/d\theta^2$ is—ve, equilibrium is unstable; when $x=\frac{4a}{3}$, $\cos\theta=\frac{29}{36}$, $\frac{d^2V}{d\theta^2}$ is-| ve. equilibrium is stable.

Example (*v*) *An isosceles triangular lamina of angle* 2α *and height* h *rests between two smooth pegs at the same level, distant* $2c$ *apart; prove that if*

$$3c\sec\alpha<h<6c\operatorname{cosec}2\alpha$$

then oblique positions of equilibrium exist, which are unstable. Discuss the stability of the vertical position.

Let OAB be the triangle, G its centre of gravity and θ the angle which OG makes with the horizontal line PQ.



Draw OL ($=p$) perpendicular from O on QP, then

$\angle OPL=\theta+\alpha$ and $\angle OQL=\theta-\alpha$,

and $2c=PQ=LQ-LP$

$=p\cot(\theta-\alpha)-p\cot(\theta+\alpha)$

$=p\sin 2\alpha/\sin(\theta+\alpha)\sin(\theta-\alpha)$,

i.e., $p=OL=c(\cos 2\alpha-\cos 2\theta)\operatorname{cosec}2\alpha$.

Now $\bar{z}$ = height of G above $PQ=OG\sin\theta-OL$

$$=\frac{2}{3}h\sin\theta-c(\cos 2\alpha-\cos 2\theta)\operatorname{cosec}2\alpha.$$

$$\therefore\quad\frac{d\bar{z}}{d\theta}=\frac{2}{3}h\cos\theta-2c\sin 2\theta\operatorname{cosec}2\alpha$$

$$=2\cos\theta\left(\tfrac{1}{3}h-2c\operatorname{cosec}2\alpha\sin\theta\right)$$

For equilibrium, this $=0$ *i.e.*, either $\theta=\frac{\pi}{2}$

$$\text{or}\quad\sin\theta=\frac{h\sin 2\alpha}{4c}$$

$$\frac{d^2\bar{z}}{d\theta^2}=-2\sin\theta\left(\tfrac{1}{3}h-2c\operatorname{cosec}2\alpha\sin\theta\right)-4c\cos^2\theta\operatorname{cosec}2\alpha. \quad (1)$$

When $\theta=\frac{\pi}{2}$, $\frac{d^2\bar{z}}{d\theta^2}$ is positive if $h<6c$ cosec $2a$, so that the vertical position of equilibrium is stable if

$$h<6c \text{ cosec } 2a.$$

For the oblique position of equilibrium since

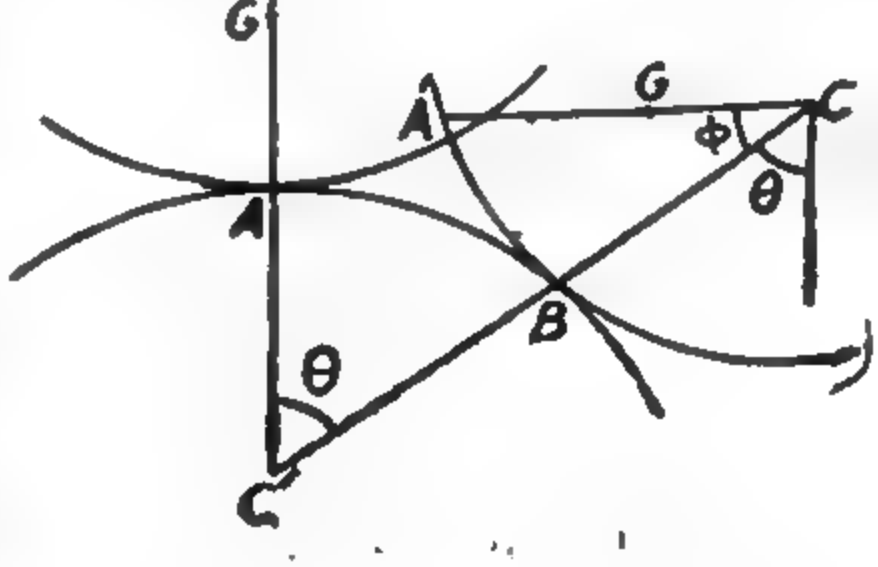
$\sin\theta=\frac{h \sin 2a}{6c}=\frac{h}{6c \text{ cosec } 2a}$ the first term in (1) vanishes and the second term is evidently negative, so that this position of equilibrium is unstable.

Now since $\sin\theta<1$ $\therefore$ $h<6c$ cosec $2a$. Also from the figure, since $\theta>a$, $\sin\theta>\sin a$

$\therefore$ $h \sin 2a>6c \sin a$ i.e., $h>3c \sec a$.

Example. (*vi*) *A rocking stone rests on a fixed stone, the contact being rough, and the common normal at the point of contact being vertical. If ρ, ρ' be the radii of curvature of the surfaces of the two stones at the point of contact, and if h be the height of the C. G. of the movable stone, show that the equilibrium of the rocking stone is stable if* $\frac{1}{h}>\frac{1}{\rho}+\frac{1}{\rho'}$.

Let C, C′ be the centres of curvature of the two stones at the point A, G the C. G. of the movable stone, where AG$=h$.



In the position of equilibrium, CC′, the common normal is vertical, A is the point of contact and G lies on AC.

Let the upper stone be slightly displaced by rolling (without sliding), so that A now occupies the position A′ and B is the point of contact.

In this position, let the common normal C′BC, and A′C make angles θ and $\theta+\phi$ respectively with the vertical.

Now since the arc A′B$=$AB, C′B$=\rho'$, BC$=\rho$,

$$\rho'\theta=\rho\phi \qquad \text{.....(1)}$$

If z be the height of G above C′, we have

$z = C'C \cos\theta - CG \cos(\theta+\phi)$

$= (\rho+\rho')\cos\theta - (\rho-h)\cos(\theta+\phi)$

$\because\ CG = AC - AG = \rho - h$

$= (\rho+\rho')\cos\theta - (\rho-h)\cos\left(1+\frac{\rho'}{\rho}\right)\theta$ from (1)

$\therefore\ \frac{dz}{d\theta} = -(\rho+\rho')\sin\theta + (\rho-h)\left(1+\frac{\rho'}{\rho}\right)\sin\left(1+\frac{\rho'}{\rho}\right)\theta$

$\theta = 0$, is an evident position of equilibrium.

$$\frac{d^2z}{d\theta^2} = -(\rho+\rho')\cos\theta + (\rho-h)\left(1+\frac{\rho'}{\rho}\right)^2\cos\left(1+\frac{\rho'}{\rho}\right)\theta.$$

$$= -(\rho+\rho') + (\rho-h)(\rho+\rho')^2\frac{1}{\rho^2} \text{ when } \theta = 0.$$

For stability of equilibrium at $\theta = 0$, $\frac{d^2z}{d\theta^2}$ should be positive-

$$\therefore\quad (\rho+\rho')^2\frac{1}{\rho^2}(\rho-h) > (\rho+\rho')$$

i.e., $(\rho+\rho')(\rho-h) > \rho^2$ or $\rho\rho' > h(\rho+\rho')$

or $$\frac{1}{h} > \frac{1}{\rho} + \frac{1}{\rho'}.$$

Cor. (i) *When the upper stone rests on a horizontal plane* $\rho' \to \infty$ *and the condition of stability becomes* $h < \rho$. *Further if the surface of the rocking stone in contact with the horizontal plane is spherical of radius* r, $h < r$. (See Art. 9·7)

(ii) *If the upper stone has a plane surface in contact with a fixed sphere* $\rho \to \infty$. *Hence the equilibrium is stable if* $h < \rho'$.

EXAMPLES XXXIV

1. A uniform solid consists of a cone and a hemisphere of equal base fastened together so that the bases coincide. Show that if the vertical angle of the cone is $> 60°$ the solid will always move to an upright position if placed with the surface of the hemisphere on a horizontal plane.

2. A solid of uniform density is built up of a hemisphere of radius r, and a right circular cylinder of radius r and height h on the same circular base as the hemisphere. Show that the solid will rest in any position with the surface of the hemisphere in contact with a horizontal plane, provided $r = h\sqrt{2}$.

3. A hemispherical piece of wood, of radius a, lying on a horizontal plane is in a neutral equilibrium when a metallic disc of the same radius and thickness h is placed just over the plane portion. What is the ratio of the densities of the substances of the sphere and the disc. [*Ans.* $2h^2/a^2$

4. A solid of uniform density in the form of a portion of a paraboloid of revolution bounded by a plane perpendicular to the axis is placed with its vertex resting on a horizontal plane and with its axis vertical. Determine the height of the solid in order that equilibrium should be stable.

[*Ans.* Less than $\frac{3}{4}$th latus rectum of the generating parabola

5. A solid frustum of a paraboloid of revolution, of height h and latus rectum $4a$, rests with its vertex on the vertex of a paraboloid of revolution whose latus rectum is $4b$; show that the equilibrium is stable if

$$h < \frac{3ab}{a+b}.$$

(*D. U.* 1939)

6. Show that a rough uniform plank of thickness b, resting horizontally on the top of a circular cylinder of radius a will be in stable equilibrium if $b < 2a$.

7. Two equal uniform rods are firmly joined at one end so that the angle between them is α and they rest in a vertical plane on a smooth sphere of radius r. Show that they are in stable or unstable equilibrium according as the length of either rod is greater or less than $4r$. cosec α.

8. A uniform square board of mass M is supported in a vertical plane on two smooth pegs at the same horizontal level. The distance between the pegs is a and the diagonal of the square is $D > 4a$. If one diagonal is vertical and a mass m is attached to its lower end, prove that the equilibrium is stable, if $4am > M(D - 4a)$. (*Pub. Ser. Com.* 1938)

9. A small ring P of mass m, slides on a smooth circular wire of radius a in a vertical plane. A light string, attached to P passes over a small smooth peg at O, the highest point of the wire, and a particle Q of mass $3m$ is suspended from the other end of the string. Prove that when P is vertically below O, the system is in unstable equilibrium. (*Bm. U.*)

10. A bead of mass m slides upon a smooth circular hoop fixed in a vertical plane and is attached to a particle of mass $2m$ by a light inextensible string which passes through a smooth ring fixed mid-way between the centre of the hoop and its highest point, the heavier particle hanging vertically below the ring. Find the positions of equilibrium and determine which are stable and which are unstable.

[*Ans.* Inclination of radius to vertical 0, π, both stable; $\cos^{-1}\frac{1}{4}$, unstable.

11. A bead of weight w is threaded on a smooth circular wire of radius a which is fixed in a vertical plane. The bead is attached by a light elastic string of unstretched length $\frac{a}{2}$ and modulus λ to a fixed point vertically above, and at a distance $2a$ from the centre of the wire. Given that $\frac{3}{16}w > \lambda < \frac{1}{8}w$, show that the bead can rest in equilibrium in four different positions and investigate the stability of the equilibrium when the bead is at the highest and the lowest points of the wire. (*H.C.*)

[*Ans.* Unstable.]

12. A uniform rod of weight W and length $2a$, can turn freely about one end A, which is fixed. A light inextensible string attached to the other end passes through a smooth ring fixed at a point C, distant $2a$ from A and at the same level as A and carries at its other end a weight $\frac{1}{13}$W. When the rod is inclined at an angle θ to the horizontal and below C, show that the system is in stable equilibrium when $8\cos\theta = 1$. (*L. U.*)

13. A rod AB of length $2a$ is pivoted to a wall at A and its end B is connected by an elastic string of natural length l and modulus λ to a point C at a height $2a$ vertically above A. Prove that if $l(W+\lambda) < 4a\lambda$, there are two positions of equilibrium of which one in which $\angle BAC = \pi$ is unstable, and the

other is stable ; but if this condition is not satisfied, equilibrium is possible only when $\angle BAC = \pi$ and it is stable. (*I.S.E.*)

14. Three equal and uniform bars AB, BC, CD, each of length $2a$, are smoothly jointed at B, C and rest with BC horizontal and AB, CD each on small smooth pegs at the same level at a distance $2(a+b)$ apart.

Show that, if $2a > 3b$, there are two positions of equilibrium, and determine which of them is stable.

If $2a = 3b$, show that there is only one position of equilibrium and that it is unstable.

15. Three equal particles repelling each other according to the nth power of the distance, are connected together by three equal elastic strings. Find the position of equilibrium and show that it is stable if $n < \frac{p}{p-a}$ where a is the unstretched and p, the stretched length of the string.

16. A uniform plate in the form of an equilateral triangle is placed in a vertical plane with two of its sides in contact with two smooth pegs on the same horizontal level. The side of the plate is a and the distance between the pegs is c Prove that the position of equilibrium with one side horizontal is stable if $a < \frac{1}{2}c$. (*Pub. Cer. Com.* 1939)

17. An isosceles triangular lamina with its plane vertical rests vertex downwards between two smooth pegs in the same horizontal line ; show that there will be equilibrium if the base ma es an angle $\sin^{-1}(\cos^2 \alpha)$ with the vertical, 2α being the vertical angle of the lamina and length of the base being three times the distance between the pegs. Is the equilibrium stable or unstable ? [*Ans.* Unstable

18. A conical plug of height h and semi-vertical α is at rest in a circular hole of radius a. Show that the vertical position of equilibrium is one of stability or instability according as $16a \gtrless 3h \sin 2\alpha$ and that the position of equilibrium when the axis is inclined to the vertical is stable if $16a < 3h \sin 2\alpha$.

19. Show that the work done by a system of co-planar

forces is equal to the increase in the potential energy of the system.

An elastic string of mass m and modulus E rests unstretched in the form of a circle of radius a. It is now acted on by a repulsive force situated in its centre whose magnitude is $\mu(\text{distance})^{-2}$. Prove that the radius of the circle when it next comes to rest is a root of the equation

$$r^2 - ar = \frac{m\mu}{\pi E}.$$

(*D.U. Hons.* 1926)

CHAPTER X

Equilibrium of Strings

I. THE CATENARY

10·1 Strings, ropes and chains frequently form part of the system of bodies with which problems in Statics are concerned. It is therefore important and even interesting to discuss the equilibrium of strings under the effect of forces acting on them. In the first instance, we consider only a particular case, namely, the discussion of the nature of the curve in which a string hangs freely between two points (not in the same vertical line) under the action of gravity alone. Such a curve is called a **Catenary** or a **Chainette**.

We shall assume that the strings and chains discussed in this chapter are *perfectly flexible i.e.*, they offer no resistance whatsoever to bending at any point. The action across any section of such a string is only tangetial to the string.

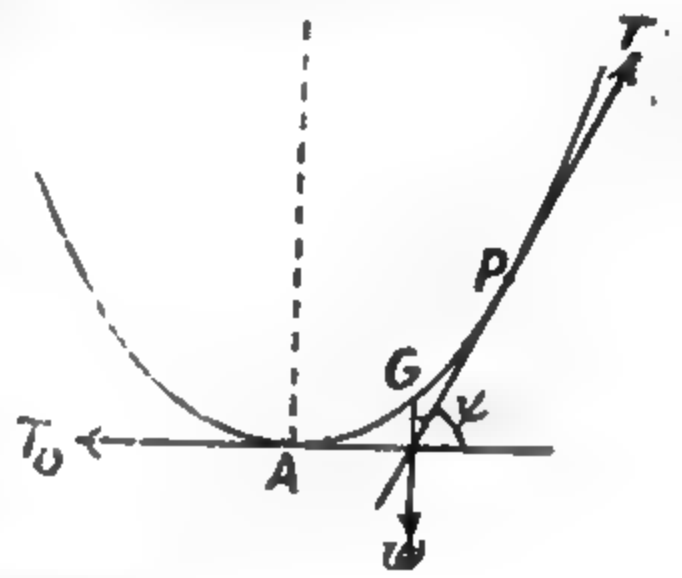
We shall also assume the normal section at every point to be so small that the form of the string may be taken as a curved line.

A chain whose links are small and perfectly smooth approximates to these ideal conditions and may be treated like a string.

A string or a chain is regarded as *uniform* when its weight per unit length is constant throughout its length.

10 11. Equation of a Catenary. *A uniform inextensible heavy string hangs freely under the action of gravity; to find the equation of the curve it assumes.*

Let A be the lowest point and P any other point on the string. Let w be its weight per unit length and ψ the angle which the tangent at P makes with the horizontal.



The forces acting on the portion $AP = s$ of the string are :—

(a) the tension T at P acting along the tangent at P.

(b) the tension T_0 at A acting horizontally.

(c) the weight, ws acting along a vertical line passing through the C. G. of the portion AP and through the point where the two tensions meet.

Resolving vertically, we have

$$T \sin \psi = ws \qquad \ldots\ldots(1)$$

Resolving horizontally, we have

$$T \cos \psi = T_0 = wc \qquad \ldots\ldots(2)$$

Where T_0 is taken to be equal to a weight of length c of the string.

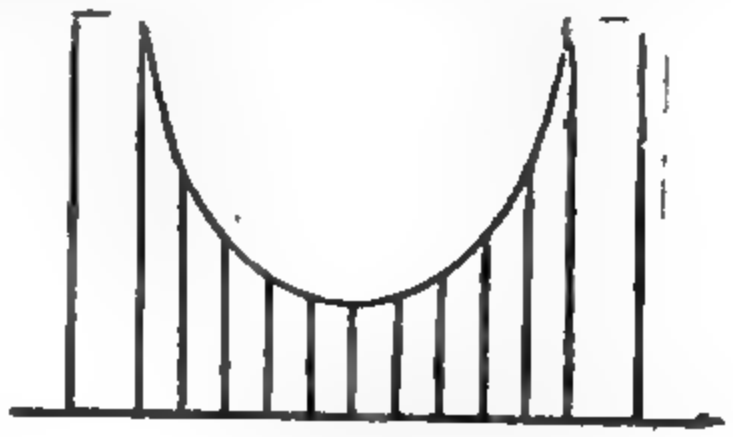
Dividing (1) by (2) we obtain $s = c \tan \psi$ .....(3)

which is the *intrinsic equation* of the catenary.

This is usually known as the common catenary, the lowest point A is known as the vertex and c is known as the parameter.

10 12. Suspension Bridge. An interesting problem is afforded by the kind of bridges (usually met with among the hills) in which the horizontal roadway is suspended from a pair of cables by a number of vertical chains, the ends of the cables being attached to the tops of rigid piers.

In this case, for the sake of simplicity, we neglect the weights of the chains and cables and suppose the weight of the bridge to be distributed evenly along its length.



Considering the equilibrium of a portion of a cable and taking the weight of each element to be proportional to its *horizontal* projection, we proceed as before.

If (x, y) be the co-ordinates of P (Fig. Art. 10·11) with reference to the horizontal and vertical lines through A, we now obtain

$$T \sin \psi = wx$$

and $$T \cos \psi = T_0 = wc$$

so that $$\frac{x}{c} = \tan \psi = \frac{dy}{dx}$$

$$\therefore \quad cy = \tfrac{1}{2}x^2 + k$$

but since $y=0$ when $x=0$ $\quad \therefore \ k=0.$

Hence the cable assumes the parabolic from given by the equation

$$\mathbf{x^2 = 2cy} \qquad \text{.....(4)}$$

The curve is therefore knowd as the **Parabolic Catenary.**

10·13. Relation between x, y and ψ.

From Calculus we have $\frac{dx}{sd} = \cos \psi$, $\frac{dy}{ds} = \sin \psi$ and from the equation $s = c \tan \psi$,

we get $$\frac{ds}{d\psi} = c \sec^2 \psi.$$

$$\therefore \quad \frac{dy}{d\psi} = \frac{dy}{ds} \cdot \frac{ds}{d\psi} = \sin \psi . c \sec^2 \psi = c \tan \psi \sec \psi$$

$\therefore$ integrating, we have, $\quad y = c \sec \psi + k$

Now we choose the axis of x, a horizontal line at a distance c below the vertex A, so that when $\psi = 0$ $y = c$ and hence $k = 0$.

$$\therefore \qquad \mathbf{y = c \sec \psi} \qquad \text{.....(5)}$$

Again $$\frac{dx}{d\psi}=\frac{dx}{ds}\cdot\frac{ds}{d\psi}=\cos\psi.c\sec^2\psi=c\sec\psi$$

$$\therefore \quad \mathbf{x=c\log(\sec\psi+\tan\psi)} \qquad \text{.....(6)}$$

the con tant of integration being zero since $x=0$, when $\psi=0$.

The horizontal line at a depth c below A which is taken as the axis of x, is known as the *directrix of the Catenary*.

When a catenary is formed by suspending the string from two points B and C in the same level, the distance BC is called the **Span** and the depth of the vertex below BC is called the **Sag**.

10 14. Cartesian equation of the catenary. This can be obtained by eliminating ψ from (5) and (6).

Form (6), we get, $e^{\frac{x}{c}}=\sec\psi+\tan\psi$

$$\therefore \quad e^{-\frac{x}{c}}=\frac{\sec^2\psi-\tan^2\psi}{\sec\psi+\tan\psi}=\sec\psi-\tan\psi.$$

Hence by addition,

$$\sec\psi=\tfrac{1}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)=\cosh\frac{x}{c}$$

But from (5) $\sec\psi=y/c$

$$\therefore \quad \mathbf{y=c\cosh\frac{x}{c}} \qquad \text{.....(7)}$$

Second Method. The same equation can also be obtained from (3) as follows :—

We have $s=c\tan\psi=c\frac{dy}{dx}=cp$, where $p=\frac{dy}{dx}$.

Differentiating with respect to x, we have

$$c\frac{dp}{dx}=\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+p^2}$$

$\therefore$ integrating, we get

$$x=\int_0^p \frac{c\,dp}{\sqrt{1+p^2}}=c\sinh^{-1}p,$$

the constant of integration being zero, for $x=0$ when $p=0$.

$$\therefore \qquad p \text{ i.e. } \frac{dy}{dx}=\sinh\frac{x}{c}$$

Integrating again, we have

$$y=c\cosh\frac{x}{c}+k.$$

But when $x=0$, $y=c$ $\therefore$ $k=0$ and $y=c\cosh\frac{x}{c}$

10·15. Relation between y and s. From equations (5) $y=c\sec\psi$ and (3) $s=c\tan\psi$, we have

$$y^2-s^2=c^2(\sec^2\psi-\tan^2\psi)=c^2$$

$$\therefore \qquad \mathbf{y^2=c^2+s^2} \qquad \text{.....(8)}$$

This relation between y and s finds frequent application in the solution of problems and can also be obtained by using (3) and (7) as follows :—

Since $y=c\cosh\frac{x}{c}$, $\therefore$ $s=c\tan\psi=c\frac{dy}{dx}=c\sinh\frac{x}{c}$,

$$\therefore\ y^2-s^2=c^2\left(\cosh^2\frac{x}{c}-\sinh^2\frac{x}{c}\right)=c^2.$$

10·2. Tension at a point.

(*a*) Since $y=c\sec\psi$ and $T\cos\psi=T_0=wc$, we have

$$\mathbf{T=wy.}$$

Thus *the tension at any point varies as its height from the directrix.*

This shows that *if a string hangs freely in equilibrinm over two smooth pegs* (not in the same vertical line), *its ends just reach the directrix of the curve between the pegs.*

(*b*) Since $T_0=wc$, it shows that T_0, the tension at the lowest point is equal to a weight of the string equal to the

height of the lowest point above the directrix, which agrees with (*a*).

(*c*) Since $T\cos\psi = T_0$, it shows that the horizontal component of the tension at every point is constant and equal to wc.

(*d*) Since by (1) Art. 10·11 $T\sin\psi = ws$, it shows that the vertical component of the tension at any point is equal to the weight of the portion of the string lying between the vertex and the point.

10.3. A tightly stretched catenary. If (x, y) be the coordinates of one end of a string stretched tightly between two points in the same horizontal level (*e. g.*, a telegraph wire between two poles), then

$$y = \cosh\frac{x}{c} = \tfrac{1}{2}c\left[e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right] = c\left[1 + \frac{x^2}{2c^2} + \frac{x^4}{24c^4} + \ldots\ldots\right]$$

$$i.e., \quad y - c = \frac{x^2}{2c} + \frac{x^4}{24c^3} + \ldots\ldots$$

Now, since the sag $y - c$ is small as compared to the span $2x$, c must be large, so that *for a tight catenary c is large and the directrix is very low.*

Shifting the origin to the vertex $(0, c)$, the equation becomes

$$y = \frac{x^2}{2c} + \frac{x^4}{24c^3} + \ldots\ldots$$

Neglecting small quantities of the third and higher orders it reduces to $y = \dfrac{x^2}{2c}$ or $x^2 = 2cy$.

Thus a *tightly stretched catenary approximates to a parabola.*

Note. When the catenary is *slack*, the lowest point of the catenary is low and consequently the directrix is low. Thus even for a slack catenary the directrix is very low because the lowest point itself is low.

10 4. Example (i) *Given two smooth pegs in a horizontal line distant 2a apart, find the least length of a uniform heavy string which will rest freely over them.*

Let $2l$ be the string $2s$ the length of the part between the pegs and y the height of a peg above the directrix.

Since the string rests freely and the pegs are smooth, by (*a*) Art. 10·2, the end of the string must reach the directix, so that half the string

$$l=y+s=c\cosh\frac{x}{c}+c\sinh\frac{x}{c}=c\,e^{x/c}=c\,e^{a/c}\quad(\because x=a)$$

$$\therefore\ \frac{dl}{dc}=e^{a/c}-c.\frac{a}{c^2}e^{a/c}=e^{a/c}\left(1-\frac{a}{c}\right).$$

For l to be least $\frac{dl}{dc}=0$ *i. e.*, $c=a$.

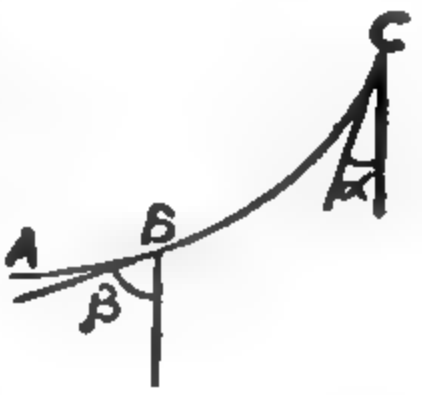
Now, $\frac{d^2l}{dc^2}=\frac{a^2}{c^3}e^{a/c}$ which is positive

$\therefore$ l is least when $c=a$ and then the least length of the string $=2l=2ce^{a/c}=2ae^{a/a}=2ae$.

Example (ii) *A heavy string of length l is supported from two points, and at these points the string makes angles α, β with the vertical. Show that the height of the one point above the other is*

$$l\cos\tfrac{1}{2}(\alpha+\beta)\sec\tfrac{1}{2}(\alpha-\beta).$$

Let the string be supported at the points B and C and let A be the vertex of the catenary of which BC forms a part.



Let y_1 y_2 be the heights of B and C above the directrix and s_1 and s_2 the lengths of the arcs AB and AC, then

$s_1=c\tan\psi=c\cot\beta$, $\quad s_2=c\cot\alpha$ and

$y_1=c\sec\psi=c\operatorname{cosec}\beta$, $\quad y_2=c\operatorname{cosec}\alpha$.

$$\therefore\ \frac{y_2-y_1}{s_2-s_1}=\frac{\operatorname{cosec}\alpha-\operatorname{cosec}\beta}{\cot\alpha-\cot\beta}=\frac{\sin\beta-\sin\alpha}{\sin(\beta-\alpha)}=\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$

Now since $s_2-s_1=l$, we have

$y_2-y_1=$the height of C above B

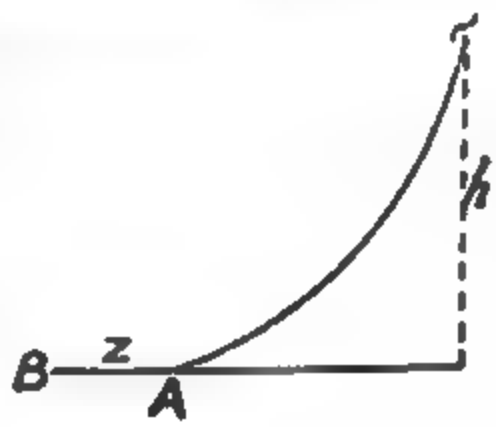
$=l\cos\frac{1}{2}(\alpha+\beta)\sec\frac{1}{2}(\alpha-\beta)$.

Example (*iii*) *A uniform chain of length l rests in a straight line on a rough horizontal table. One end is raised to a height h above the table and the chain is on the point of motion. Show that the length of the straight part on the table is*

$$l+\mu h-\{(\mu^2+1)\,h^2+2\mu lh\}^{\frac{1}{2}}$$

where μ is the co-efficient of friction.

Let z be the required length and w the weight per unit length of the chain, then since the chain is on the point of slipping, the horizontal tension at A, the lowest point of the catenary =force of friction.



i.e., $T_0=wc=\mu wz$, so that $c=\mu z$.....(*i*)

Length of the portion AP$=s=l-z$ and the height of P above the directrix

$$=y=c+h$$

$\therefore$ from the formula $y^2=c^2+s^2$, we have

$$(c+h)^2=c^2+(l-z)^2$$

Using (*i*) and simplifying, we have

$$z^2-2z(l+\mu h)+l^2-h^2=0$$

which gives $z=l+\mu h\pm\sqrt{(l+\mu h)^2-l^2+h^2}$

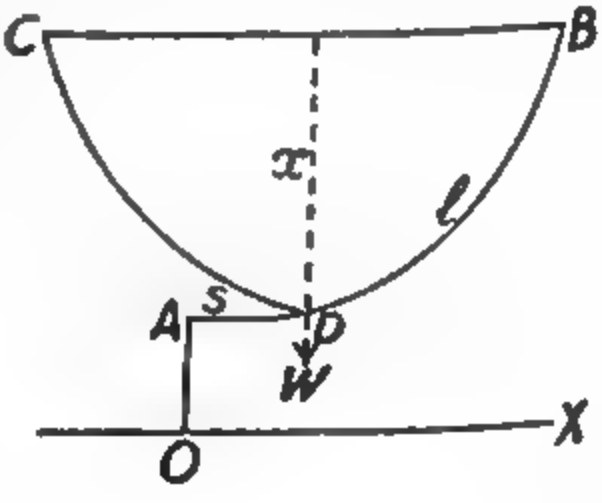
$\therefore$ Since $l-z$ must be positive, we have

$$z=l+\mu h-\sqrt{(\mu^2+1)h^2+2\mu lh}.$$

Example (*iii*) *The weight per foot run of a telegraph wire is w lb., the length of wire between the posts is 2l feet, a weight W lb. is hung at the middle, where the wire sags x feet. Prove that the tension at the posts is*

$$w\left\{\frac{l^2}{2x}\left(1+\frac{W}{lw}\right)+\tfrac{1}{2}x\right\}\text{ lb.}$$

Let B and C be the points of support of the wire, D the point of suspension of the weight, A the vertex of the catenary of which DB is a part, s the arc AD, y, y_1 the ordinates of B and D and T, T_1 the tensions at B and D.



Since the weight W is supported by both the parts CD and DB, $\frac{1}{2}$W is the vertical component of the tension at D for the part AD of the string.

$\therefore$ by (*d*) Art. 10·2, $\frac{1}{2}W = ws$ *i.e.*, $s = W/2w$ .....(1)

By (*a*) Art. 10·2, $T = wy$ and $T_1 = wy_1$

$\therefore$ $T - T_1 = w(y - y_1) = wx$ .....(2)

and $T + T_1 = w(y + y_1)$ .....(3)

But by Art. 10·15, $y^2 = c^2 + (s+l)^2$

and $y_1^2 = c^2 + s^2$

$\therefore$ $2ls + l^2 = y^2 - y_1^2 = (y - y_1)(y + y_1) = x(y + y_1)$

so that from (3) and (1), $T + T_1 = w\left[\frac{l^2}{x} + \frac{2l}{x} \cdot \frac{W}{2w}\right]$ .....(4)

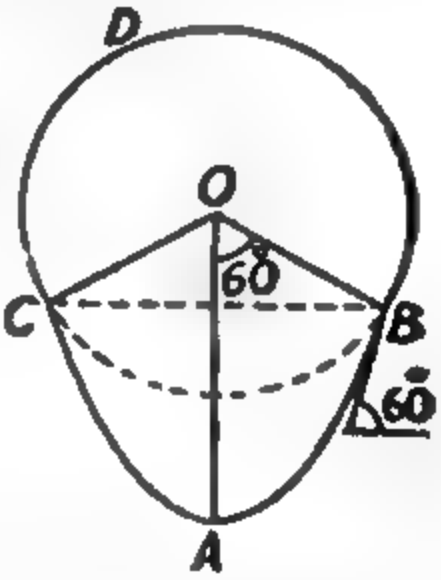
Hence from (2) and (4)

$$2T = wx + w\,\frac{l^2}{x}\left[1 + \frac{W}{lw}\right].$$

Example (*v*). *Show that the length of an endless chain which will hang over a circular pulley of radius a, so as to be in contact with two-thirds of the circumference of the pulley is*

$$a\left[\frac{3}{\log(2+\sqrt{3})} + \frac{4\pi}{3}\right].$$ (*D.U.* 1939; *Agra.* 1940)

Let ACDBA be the endless chain hung over the circular pulley CDB with centre O.



Since the chain is in contact with $\frac{2}{3}$ circumference of the pulley, the length of the part in contact with the pulley

$= \frac{2}{3} \times 2\pi a = \frac{4}{3}\pi a.$ .....(1)

Also $\angle BOC = \frac{1}{3} \times 2\pi = 120°$

and $BC = 2a \sin 60°$

$= \sqrt{3}\, a$

Now we consider the catenary CAB.

The angle which the tangent at B makes with the horizontal

$= \angle AOB = 60°$

$\therefore$ from $s = c \tan \psi$, we have

arc CAB = 2 arc AB $= 2c \tan 60° = 2\sqrt{3}\, c.$ .....(2)

To find c, using $x = c \log (\tan \psi + \sec \psi)$ we have

$\frac{1}{2}$ BC $= \frac{1}{2} \sqrt{3}\, a = c \log (\tan 60° + \sec 60°) = c \log (\sqrt{3}+2)$ .....(3)

$\therefore$ from (2) and (3), we have

the length of the catenary $= 2\sqrt{3}\, c = \dfrac{3a}{\log (\sqrt{3}+2)}$ .....(4)

Hence the total length of the string

$$= \frac{3a}{\log (\sqrt{3}+2)} + \frac{4\pi a}{3} = a\left[\frac{3}{\log (2+\sqrt{3})} + \frac{4\pi}{3}\right].$$

EXAMPLES XXXV

1. A kite is flown with 600 ft. of string from the hand to the kite and a spring balance held in the hand shows a pull equal to the weight of 300 ft. of the string inclined at 30° to the horizon. Find the vertical height of the kite above the hand.

[*Ans.* 300 ($\sqrt{7}$—1) ft.]

2. A uniform chain of length 15 ft. hangs between two points A and B, the vertical distance between them being 9 ft. It is observed that the tangent at A is horizontal. Show that the horizontal distance between A and B is

$$\left(8 \cosh^{-1} \frac{17}{8}\right) \text{ft.}$$

3. A suspension bridge of weight 100 tons evenly distributed over its span of 150 ft. is supported by two cables which have a sag of 20 ft. Find the greatest and least tensions in each cable.

[*Ans.* $53\frac{1}{8}$, $46\frac{7}{8}$ tons.]

3 (*a*) In a suspension bridge of 400 feet span and 40 feet dip the whole weight supported by the chains is 2 tons per horizontal foot. Find the horizontal tension and the tension at the points of support.

If a simple telegraph wire has a span of 75 yards, and a sag in the middle is 1 ft. show that the tension in the wire is approximately 480 lb. weight, if the weight of the wire is 400 lb. per mile.

[*Ans.* 500, 538·5 tons.]

4. A uniform chain is hung from two points A and B in the same horizontal line, and at a distance $2a$ apart. The inclination of the chain to the horizontal at A or B is α. If $2s$

be the total length of the chain, w the weight per unit length, T_0 the tension at the lowest point and d the sag, show that

$$\frac{\tan \alpha}{s} = \frac{w}{T_0} = \frac{\sec \alpha - 1}{d} = \frac{\log_e (\tan \alpha + \sec \alpha)}{a}. \qquad (M.\ T.)$$

5. A uniform chain, of weight W and length $2l$ hangs between two point A, B in the same horizontal line at a distance $2a$ apart, and the sag in the middle is d. Show that the parameter c of the catenary is $\frac{l^2-d^2}{2d}$ and that the tension at either end is $W(l^2+d^2)/4ld$.

6. A rope of length $2l$ ft. is suspended between two points at the same level, and the lowest point of the rope is b ft. below the point of suspension. Show that the horizontal component of the tension is

$$\frac{w}{2b}(l^2-b^2),$$

w being the weight of tho rope per foot of its length.

7. The end links of a uniform chain of length l can slide on two smooth rods in the same vertical plane which are inclined in opposite directions at equal angles α to the horizontal. Prove that the sag in the middle is

$$\tfrac{1}{2} l \tan \tfrac{1}{2} \alpha.$$

8. A uniform heavy chain of length $2l$ has its extremities fastened to two points at the same level and the sag in the middle is h. Find the parameter of the catenary in which it hangs and prove that the span is

$$\frac{l^2-h^2}{h} \log \frac{l+h}{l-h}.$$

9. A uniform chain is hung up from two points at the same level and distant $2a$ apart. If z is the sag at the middle, show that

$$z = c\left(\cosh \frac{a}{c} - 1\right).$$

If z be small compared with a, show that $2cz=a^2$ nearly.

A telegraph wire is supported by two poles distant 40 yards apart. If the sag be one foot and the weight of the wire $\frac{1}{8}\frac{1}{2}$ lb. per ft., show that the horizontal pull on each pole is $\frac{1}{2}$ cwt. nearly.

10. A uniform measuring chain of length l, is tightly stretched over a river, the middle point touching the surface of the water, while each of the extremities has an elevation k above the surface, show that the difference between the length of the measuring chain and the breath of the river is nearly $\frac{8}{3}\frac{k^2}{l}$. *(D. U. 1924)*

11. A heavy chain rests in equilibrium in a vertical plane supported by two smooth pegs A and B. If α and β are the angles the tangents to the chain at A and B make with the vertical and if x and y are the lengths of the chain hanging vertically from A and B, and z is the length between A and B, show that

$$\frac{x}{\sin\beta}=\frac{y}{\sin\alpha}=\frac{z}{\sin(\alpha+\beta)}.$$

12. One end of a perfectly flexible uniform chain, of length l, is attached to a fixed point at height h above a smooth horizontal table and a horizontal distance b from the edge over which the chain hangs. The chain consists of three parts, namely, a catenary of length s, a straight part of length $b-x$ on the table (perp to the edge) and a straight part hanging vertically. Prove that

$$s+b=\sqrt{2b(x+l)}.$$

13. A weight an nth that of the string is attached to its lowest point, the tengents at the lowest and the highest points of the string are now inclined to the vertical at angles ϕ and θ respectively ; show that

$$\tan\phi=(1+n)\tan\theta. \qquad (D.U.\ 1938)$$

14. A uniform chain of length $2l$ and weight $2wl$ hangs from two fixed points on the same level and at a distance $2a$ apart. A smooth ring of weight W is threaded on the chain.

Show that c, the parameter of either catenary formed by the chain can be determined from the equation

$$2wl = W\left(\cosh\frac{a}{c} - 1\right) + (W^2 + 4w^2c^2)^{\frac{1}{2}} \sin h\frac{a}{c}.$$

15. A uniform chain of weight w per unit length, to which a weight W is attached at the middle point O, hangs symmetrically over two smooth pegs on the same horizontal level and distant $2d$ apart. The chain is of such a length that, when the system is in equilibrium a length l of the chain hangs vertically over each peg, and the tangent at O to each half of the chain makes an angle α with the horizontal.

If O is at a vertical distance D below the pegs and $W = 2wc \tan\alpha$, prove that

(*i*) $D = l - c \sec\alpha$.

(*ii*) $\frac{l}{c} = \sec\alpha \cosh\frac{d}{c} + \tan\alpha \sinh\frac{d}{c}$. (*D. U.* 1937)

16. A uniform chain of length l is suspended from two points A, B in the same horizontal line. If the tension at A is twice that at the lowest point, show that the span AB is

$$\frac{l}{\sqrt{3}} \log(2+\sqrt{3}).$$

17. A uniform chain of length l, is to be suspended from two points, A and B, in the same horizontal line so that either terminal tension is n times that at the lowest point. Show that the span AB must be

$$\frac{l}{\sqrt{n^2-1}} \log(n+\sqrt{n^2-1}).$$ (*Agra.* 1941)

18. A uniform chain of length 16 ft. lies on a rough floor, (the co-efficient of friction being $\frac{1}{4}$), in a straight line. One end is lifted up to a height of 4 ft., and the chain is then on the point of slipping along the floor. Prove that the length of the chain on the floor is 10 ft.

19 A uniform inextensible string of length l and weight W, hangs freely under gravity. Its ends can slide on a fixed

rough horizontal rod whose co-efficient of friction is μ, show that at most they can rest apart at a distance

$$\mu l \log\left(\frac{1+\sqrt{1+\mu^2}}{\mu}\right)$$ (*Agra.* 1938)

20. A horizontal table has a circular hole in it of diameter one foot. A uniform chain 5 ft. long is laid symmetrically across the hole, and is found to rest in limiting equilibrium when 2 ft of the chain hangs through the hole. Prove that μ, the co-efficient of friction between the chain and the table can be found by solving the equation.

$$\text{Sinh } u = 2u, \qquad \text{where} \qquad u = 1/\sqrt{9\mu^2 - 4}.$$

21. A uniform chain hangs in equilibrium under gravity with one end attached to a fixed point O, the chain passes over a smooth peg P, O and P are in the same level and at distance $2a$ apart. If the length of the chain is the smallest possible, show that c, the parameter of the catenary formed by the curved part of the chain is given by the equation,

$$e^{2x} = \frac{1+x}{1-x} \qquad \text{where } x = a/c.$$ (*H. O.*)

22. A weight W is suspended from a fixed point by a uniform string of length l and weight w per unit of length. It is drawn aside by a horizontal force P. Show that in the position of equilibrium, the distance of W from the vertical through the fixed point is

$$\frac{P}{w}\left[\sinh^{-1}\left(\frac{W+lw}{P}\right) - \sinh^{-1}\left(\frac{W}{P}\right)\right].$$ (*Nagpur.* 1937)

What happens when the string is light ?

23. A uniform chain of length l hangs between two points A and B which are at a horizontal distance a from one another, with B at a vertical distance b above A. Prove that the parameter of the catenary is given by

$$2c \sinh \frac{a}{2c} = \sqrt{l^2 - b^2}.$$

Prove also that, if the tensions at A and B are T_1 and T_2, respectively,

$$T_2 - T_1 = \frac{Wb}{l} \quad T_1 + T_2 = W\sqrt{1 + \frac{4c^2}{l^2 - b^2}}$$

where W is the weight of the chain (*Birmingham* 1938)

II. Strings or Chains on a Plane Curve.

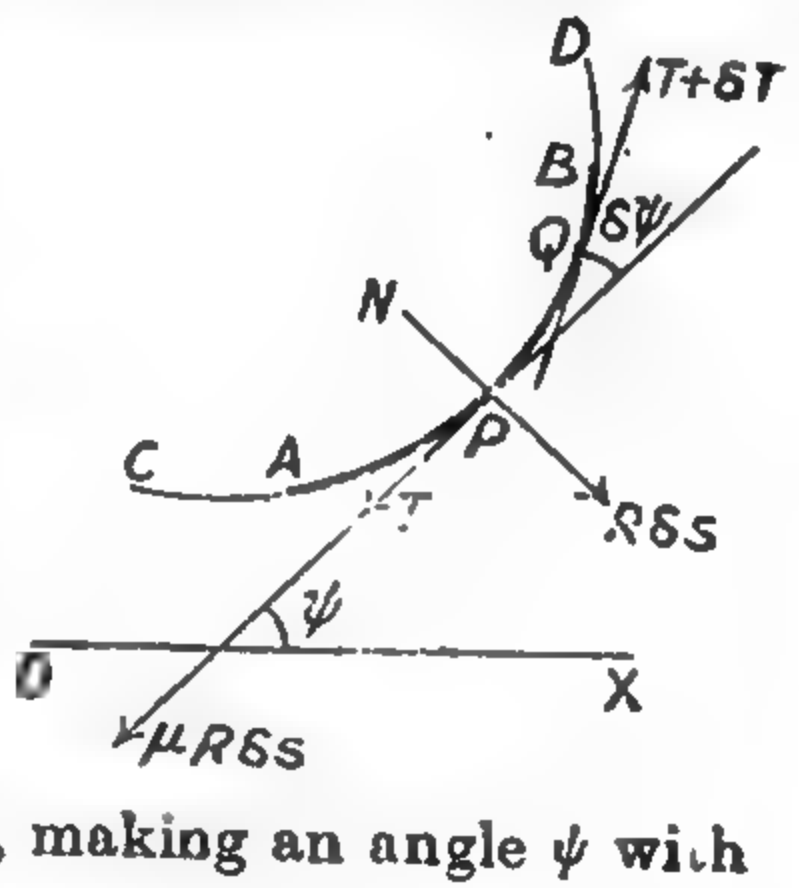
10·5. In Art. 10·11, we dealt with the equilibrium of a string, hanging freely under gravity. We now consider the equilibrium of a *light or heavy* string resting in contact with a *smooth or rough* plane curve (or resting in one plane on a surface).

10·51. Case I. Light *inextensible uniform string resting on a* **smooth** *plane curve.*

Let a string CABD be in contact with the plane curve AB and let us consider the equilibrium of a small element PQ of the string.

Let s be the length of the string measured from some fixed point C, upto a variable point P, and δs, the length of the element PQ.

Let R be the reaction per unit length of the curve at P. Then the string being light and the curve smooth the forces acting on the eiement are :—



(1) Tension T along the tangent P, making an angle ψ with a fixed line OX.

(2) Tension $T+\delta T$ along the tangent at Q making an angle $\psi+\delta\psi$ with OX.

(3) $R\delta x$, the reaction of the curve along the normal NP.

The string may either be resting on the upper side of the curve, the reaction acting inwards, or, as shown in the figure, may press against the lower side of the curve the reaction acting outwards along the normal at P.

The force $\mu R\delta s$, shown in the figure, is not to be considered in this case.

Resolving the forces along the tangent and normal at P, we have

$$(T+\delta T)\cos\delta\psi = T \qquad \ldots\ldots(1)$$

and $$(T+\delta T)\sin\delta\psi = R\delta s. \qquad \ldots\ldots(2)$$

Neglecting small quantities of higher order than the first, these equations become

$$\delta T = 0 \qquad i.e., \qquad T = \text{a constant.}$$

so that *the tension of a light string passing round a smooth curve is constant throughout.*

and from (2) $T\delta\psi = R\delta s$ *i.e.*, $R = T\frac{d\psi}{ds} = \frac{T}{\rho}$

T, being constant, the normal reaction varies inverseiy as the radius of curvature of the curve.

10·52. Case II. Light *string resting in limiting equilibrium on a* **rough** *plane curve.*

As in the previous case, let $PQ = \delta s$ be an element of the string and let the angles that the tangents at P and Q make with the fixed line OX be ψ and $\psi+\delta\psi$.

Let the tensions at A and B where the string leaves the curve be T_1 and T_2 and let the string AB be just on the point of slipping from A towards B, so that the friction $\mu R\delta s$ acts along the tangent PT, opposite to the direction of slipping.

Resolving the forces along the tangent and normal at P, we have

$$(T+\delta T)\cos\delta\psi = T+\mu R\delta s \qquad \ldots\ldots(1)$$

and $$(T+\delta T)\sin\delta\psi = R\delta s \qquad \ldots\ldots(2)$$

Neglecting small quantities of higher order than the first, we get

$$\delta T = \mu R\delta s \qquad \text{and} \qquad T\delta\psi = R\delta s$$

which give $\frac{dT}{T} = \mu d\psi$ *i.e.* $\log T = \mu\psi + a$ const. or $T = ce^{\mu\psi}$

Now, at the point A, let $T=T_1$ and $\psi=\psi_1$ and at B let $T=T_2$ and $\psi=\psi_2$, then

$$T_1=ce^{\mu\psi_1} \text{ and } T_2=ce^{\mu\psi_2} \quad \therefore\ T_2=T_1{}^{\mu(\psi_2-\psi_1)}$$

which gives the tension at B in terms of that at A and the angle the tangent turns through from A to B.

For example, if a rope is passed round a post through one complete revolution, the tension is increased in the ratio $e^{2\pi\mu}$ If $\mu=\cdot5$, $2\pi\mu=\pi=3\ 1416$ and $e^{\pi}=(2\cdot718)^{3\cdot1416}=$ about 23, so that the tension is increased about 23 times for each complete turn.

Example. *A light string passing over a rough horizontal beam, carries a weight of 80 lb. at one end. It is found that a weight of 20 lb when tied to the other end just prevents slipping. What would be the least weight necessary to prevent slipping, if the string were wouned an additionul time round the beam ?*

Here $T_2=80$, $T_1=20$, $\psi_2-\psi_1=\pi$. $e^{\mu\pi}=\frac{80}{20}=4$.

Let w be the required least weight when the angle is 3π, then

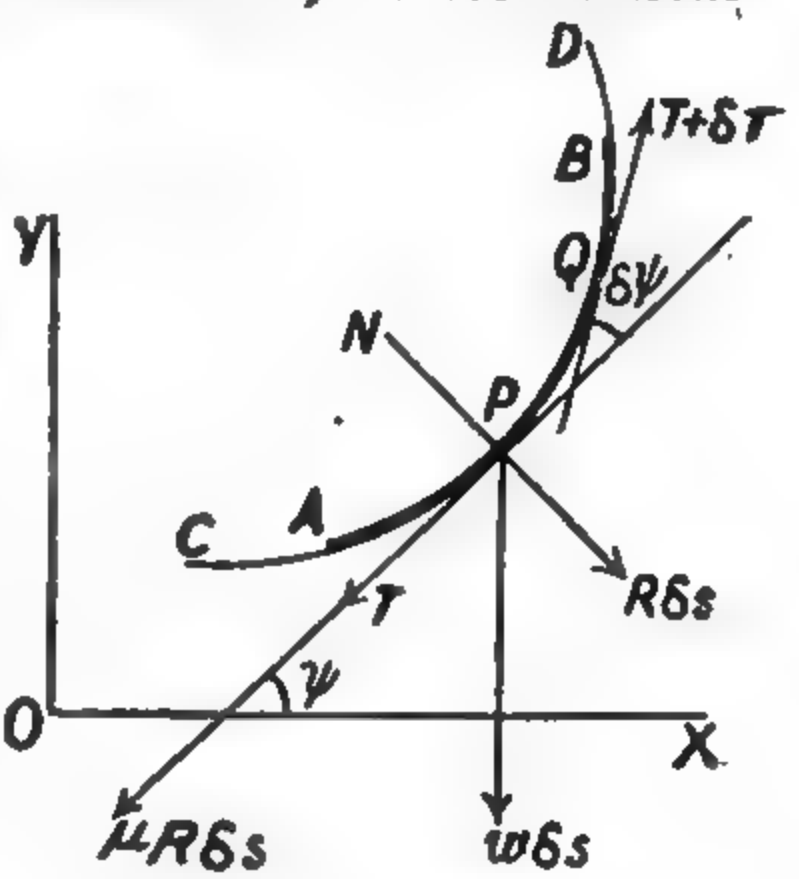
$$\frac{80}{w}=e^{3\mu\pi}=(4)^3=64 \qquad i.e., \qquad w=80/64=1\tfrac{1}{4}\text{ lb.}$$

10·53. Case III. Heavy *string resting on a* **smooth** *curve in a vertical plane.*

In addition to the tensions at P and Q and the normal reaction considered in Case I, in this case we have also to take into account a vertical force $w\delta s$ corresponding to the weight of the element PQ, w being the weight per unit length of the string.

The force $\mu R\delta s$, shown in the figure, is not to be considered in this case.

Let the line OX, from which ψ is measured, be horizontal and let it be taken as the axis of x. Let (x, y) be the co-ordinates of P, the axis of y being vertically upwards.



Then, resolving the forces along the tangent and normal, we have

$(T+\delta T)\cos\delta\psi = T + w\delta s \sin\psi$ .....(1)

and $(T+\delta T)\sin\delta\psi$

$= R\delta s + w\delta s\cos\psi$ .....(2)

Neglecting small quantities of higher order than the first, we have

$$\delta T = w\delta s \sin\psi = w\delta s \frac{\delta y}{\delta s} = w\delta y \quad i.e., \quad T = wy + C \text{.....(3)}$$

and $$R = T\frac{d\psi}{ds} - w\cos\psi = \frac{T}{\rho} - w\cos\psi \text{.....(4)}$$

Now if T_1, T_2 be the tensions at points where the ordinates are y_1 and y_2, then from (3), $T_1 - T_2 = w(y_1 - y_2)$.

i.e., if a heavy string rests against a smooth plane curve, the difference of the tensions at two points is equal to the weight of a length of the string equal to the vertical distance between the points.

Example 1. *If a heavy uniform string passes round various smooth curves in the same vertical plane and its ends hang vertically, show that they are in the same horizontal straight line.*

Since the ends hang freely, $T_1 = T_2 = 0$.

$\therefore \quad y_1 - y_2 = 0 \quad i.e., \quad y_1 = y_2$,

so that ends of the string, even when it passes over a number smooth curves, hang vertically at the same level.

Example 2. *A heavy uniform chain of weight w lb. per unit length rests on a smooth circular arc, radius a, in a vertical plane. The chain occupies a quadrant with one end at the highest point and is kept in equilibrium by a force F at the upper end, applied horizontal, find F.*

Here the vertical distance between the ends is equal to the radius a, and the tension at the lower end is zero.

$\therefore$ the tension at the highest point along the horizontal tangent $=wa$. Hence $F=wa$.

Example 3. *A heavy uniform chain rests symmetrically on a smooth catenary whose axis is vertical and vertex upwards; find the tension at any point and show that the pressure on the curve varies inversely as the square of the distance below the directrix.*

The axis of y being downwards, there is a change of sign in the weight and equations (3) and (4) become in the case

$$T=C-wy \qquad \text{and} \qquad R=T/\rho+w\cos\psi.$$

In the case of a catenary, we have $s=c\tan\psi$,

$$y=c\sec\psi \quad \text{and} \quad \rho=c\sec^2\psi.$$

At the lowest point of the chain where $T=0$, let $\psi=\psi_0$.

$\therefore\ T=C-wy=C-wc\sec\psi$ and $0=C-wc\sec\psi_0$.

$\therefore\qquad T=wc(\sec\psi_0-\sec\psi)$

Which gives the tension at any point.

$$\text{Also } R=\frac{T}{\rho}+w\cos\psi=\frac{wc(\sec\psi_0-\sec\psi)}{c\sec^2\psi}+w\cos\psi$$

$$=\frac{w\sec\psi_0}{\sec^2\psi}=\frac{wc^2\sec\psi_0}{y^2}.$$

10·54. Case IV. Heavy *string resting in limiting equilibrium on a* **rough** *curve in a vertical plane.*

In this case besides the weight $w\delta s$, we have also to take into account the force of friction $\mu R\delta s$.

In place of equation (1) and (2) of the previous article, we now have

$$(T+\delta T)\cos\delta\psi=T+\mu R\delta s+w\delta s\sin\psi \qquad \ldots\ldots(1)$$

and

$$(T+\delta T)\sin\delta\psi=R\delta s+w\delta s\cos\psi \qquad \ldots\ldots(2)$$

Neglecting small quantities of higher order than the first, we have

$$\delta T=\mu R\delta s+w\delta s\sin\psi \text{ and } T\delta\psi=R\delta s+w\delta s\cos\psi.$$

Eliminating R, dividing by $\delta\psi$, and putting $ds/d\psi=\rho$, we get $$\frac{dT}{d\psi}-\mu T=w\rho(\sin\psi-\mu\cos\psi) \quad \text{.....(3)}$$

Multiplying by $e^{-\mu\psi}$ and integrating, we get

$$Te^{-\mu\psi}=C+w\int e^{-\mu\psi}\rho(\sin\psi-\mu\cos\psi)d\psi \text{...(4)}$$

For the known curve on which the string rests, we can obtain ρ in terms of ψ and can find T by calculating the valuo of the integral.

Example. *A heavy uniform chain, resting on a rough circular arc in a vertical plane, occupies a quadrant with one end at the highest point A and is on the point of slipping. Prove that the co-efficient of friction μ satisfies the equation.*

$$(1-\mu)e^{\mu\frac{1}{2}\pi}=2\mu$$

Let AX the tangent at A be the axis of x and let the axis of y be downdwards. Let the tangent at any p.int P make an angle θ with AX, then changing the sign of w in (4) and putting $\psi=\theta$ and $\rho=a=$radius of the circle, we have

$$Te^{-\mu\theta}=C-wa\int e^{-\mu\theta}(\sin\theta-\mu\cos\theta)d\theta \quad \text{..... (1)}$$

$$=C-wa\left[-\frac{e^{-\mu\theta}(\cos\theta+\mu\sin\theta)}{1+\mu^2}-\frac{\mu e^{-\mu\theta}(\sin\theta-\mu\cos\theta)}{1+\mu^2}\right]$$

$$=C+wa\frac{e^{-\mu\theta}}{1+\mu^2}[\cos\theta+2\mu\sin\theta-\mu^2\cos\theta]$$

Now when $\theta=0$, $T=0$ $\therefore$ $0=C+\frac{wa}{1+\mu^2}[1-\mu^2]$

and when $\theta = \frac{\pi}{2}$, $T=0$ $\therefore$ $0=C+\frac{wa}{1+\mu^2}e^{-\frac{1}{2}\mu\pi}$ [2μ].

Eliminating C, and simplifying, we get

$$\frac{e^{\frac{1}{2}\pi\mu}}{(1-\mu^2)}=2\mu.$$

Note. The student is advised to draw the figure and obtain equation (1) independently.

10·6. *Equations of equilibrium of a string in one plane under the action of any given forces.*

As in previous articles, we consider the equilibrium of an element PQ of the string taking tensions T and $T+\delta T$ at its ends and resolving the forces along the tangent and normal at P.

If F per unit length be the sum of the components of the external forces, along the tangent at P in the sense PQ and G the component along the outward normal, the equations of equilibrium are

$$(T+\delta T)\cos\delta+\psi F\delta s=T \quad \ldots\ldots(1)$$

and
$$(T+\delta T)\sin\delta\psi=G\delta s \quad \ldots\ldots(2)$$

Neglecting small quantities of higher order than the first, we have

$$\delta T+F\delta s=0 \quad \text{and} \quad T\delta\psi=G\delta s \quad \ldots\ldots(3)$$

These can be written as

$$\frac{dT}{ds}+F=0 \quad \text{and} \quad \frac{T}{\rho}-G=0.$$

These are the general equations and all the four previous articles can be deduced from them as particular cases.

Note. Comparing equations (1) and (3) with (3) it may be seen that as far as tensions at the ends of the element are concerned, their resultant reduces to δT, in the ψ increasing direction of the tangent and $T\delta\psi$ in the direction of the outward normal at P. In examples that follow, these expressions may be used and equations written down independently without using the results of the previous articles as formulae.

$T\delta\psi$ may be written as $T\frac{\delta\psi}{\delta s}\delta s=T\frac{d\psi}{ds}\delta s=\frac{T}{\rho}\delta s$.

EXAMPLES XXXVI

1. Masses of 50 lb. and 80 lb. are connected by a light string which passes over a rough cylindrical beam and the system is in limiting equilibrium. What mass must be added to the 50 lb. in order that the other mass may be on the point of moving upwards. [*Ans.* 78 lb.

2. Two rough cylindrical beams have their axes horizontal on the same horizontal level and parallel, at a distance d apart. The radii are a, b and the co-efficients of friction μ, μ_1. Weights W and W_1 are connected by a light inelastic string which passes over both the beams. Show that the system will remain in equilibrium provided that

$$\log_e W/W_1 \text{ lies between } \pm\left[\pi\mu-(\mu-\mu_1)\cos^{-1}\left(\frac{a-b}{d}\right)\right]$$

3. A single movable pulley, of weight W, is just supported by a force F which is applied to one end of a light cord which goes under the pulley and is then fastened to a fixed point, show that if θ be the angle subtended at the centre by the part of the string in contact with the pulley, then

$$F^2(1-2e^{\mu\theta}\cos\theta+e^{2\mu\theta})=W^2.$$

μ, being the co-efficient of friction between the cord and the pulley. Deduce the case when the pulley is smooth.

4. A heavy particle is attached to an endless light inextensible string which passes over a rough pulley fixed in a vertical plane. If the straight parts of the string are inclined to each other at an angle α, prove that the angle θ which one of the parts makes with the vertical in the limiting position of equilibrium is given by

$$\cot\theta=\cot\alpha+e^{\mp(\mu\pi+\alpha)}\operatorname{cosec}\alpha.$$

5. A uniform heavy chain passes round a smooth vertical circle. If the pressure vanishes at the lowest point, prove that the tension (*i*) at the highest point is three times and (*ii*) at the ends of the horizontal diameter twice of what it is at the lowest point.

6. A uniform heavy string of weight w per unit length, rests on a smooth parabola, whose axis is vertical and vertex

upwards, so that its ends are at the extremities of the latus-rectum. Show that the pressure on the curve, at the point where the tangent makes an angle θ with the vertical is

$$\tfrac{1}{2}w(2\sin^3\theta+\sin\theta).$$

7. A uniform string rests on a smooth cycloidal curve, $s=4a\sin\psi$, whose axis is vertical and vertex upwards, the ends of the string just reaching the cusps of the cycloid. Prove the pressure at any point varies as the radius of curvature.

8. A uniform string of weigth w per unit length passes under the arc of a smooth cycloid, with vertex downwards and axis vertical, and over two smooth pegs at the cusps. If the string is to remain in contact at every point of the cycloid, show that its length must not be less than $20a$, a being the radius of the generating circle.

9. A uniform heavy chain rests on the upper surface of a rough vertical circle of radius a and partly hangs freely. If one end be at the highest point of the circle, prove that the greatest length that can hang freely is

$$2\mu a+(\mu^2-1)a\,e^{\frac{1}{2}\mu\pi}\ ]\ /\ (1+\mu^2).$$

10. A uniform string of weight w per unit length lies on the upper half of a horizontal circular cylinder of radius a cutting the generators of the cylinder at right angles. Show that, if $\mu=\tan\phi$ be the co-efficient of friction, the downward force at one of the ends which make the string slip is

$$aw(1+e^{\mu\pi}\ )\sin 2\phi.$$

11. A heavy uniform chain rests on the rough cycloid $s=4a\sin\psi$ in a vertical plane, vertex upwards and axis vertical. One end of the chain is at the vertex and the other at a cusp. Show that

(1) if the equilibrium be limiting, the co-efficient of friction μ is given by

$$(1+\mu^2)e^{\frac{1}{2}\mu\pi}=3.$$

(2) the least horizontal force applied at the upper end that will cause the string to slip upwards is

$$W(3e^{\frac{1}{2}\mu\pi}-\mu^2-1)\ /\ (\mu^2+4)$$

where W is the weight of the string.

CHAPTER XI

FORCES IN THREE DIMENSIONS.

11·1 Specification of a couple. In Art. 5·5, it was shown that couples in the same plane or in parallel planes can be compounded into a single couple whose moment is equal to the sum of the moments of the various couples and whose axis is a line perpendicular to the planes of the couples.

We can therefore represent a couple by a straight line (say, AB) of a certain length drawn in a direction parallel to the axis of the couple, then

(*i*) the length of the line indicates the magnitude of the moment of the couple, on a certain scale,

(*ii*) the direction of the line indicates the plane of the forces, the plane being perpendicular to the line and

(*iii*) the order of the letters AB or BA, indicates the sense of rotation of the couple.

11·11. Since a couple can be specified by a straight line drawn in a certain direction it shows that it is a *vector* quantity. It follows therefore, that couples can be compounded or resolved by the parallelogram law in the same way as forces.

Thus if L, M, N be the moments of the couples about rectangular axes OX, OY, OZ, their resultant G is given by

$$G^2 = L^2 + M^2 + N^2$$

the direction cosines of the axis of G being $\frac{L}{G}$, $\frac{M}{G}$ and $\frac{N}{G}$.

11·12. Convention of sign. If OX, OY, OZ be a set of three mutually perpendicular axes a couple in the plane XOY (or in a plane parallel to it) will be represented by a line of certain length taken along OZ (or along a line parallel to OZ).

If the couple causes rotation from OX towards OY, the moment of the couple is regarded as positive and it is then

represented by a positive length along OZ. Similarly a couple in the plane YOZ is considered positive if it causes rotation from OY towards OZ and a couple in the plane ZOX is positive when rotation is from OZ towards OX.

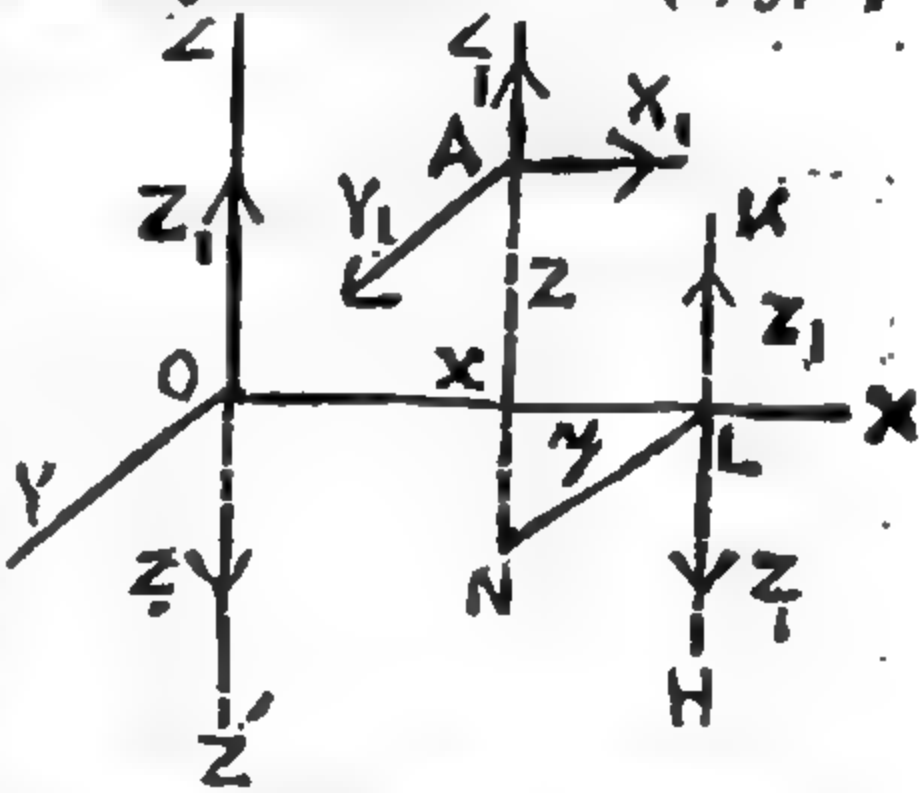
11·2. *To prove that a system of non-coplanar forces acting on a rigid body can be reduced in general to a force acting through a specified point of the body and a couple.*

Let O, the specified point be taken as the origin and let OX, OY, OZ be a set of rectangular axes through O.

Let one of the forces P of the system act at A (x, y, z) and let the components of P parallel to the axes be X_1, Y_1, Z_1.

Draw AN perp. on the XOY plane and NL perp. to OX so that $OL = x$, $NL = y$ and $NA = z$

At L and at O introduce equal and opposite forces equal and parallel to Z_1. These being in equilibrium among themselves, do not alter the effect of the system.



Now the forces Z_1 along NA and LH form a couple of moment $+yZ_1$ in the sense from OY to OZ.

Also the forces Z_1 along LK and OZ′ form a couple of moment $-xZ_1$ in sense from OX to OZ.

We have also a force Z_1 along OZ. Hence the force Z_1 at A is equivalent to

(i) an equal and parallel force Z_1 along OZ,
(ii) a couple of moment $+yZ_1$ about OX,
(iii) a couple of moment $-xZ_1$ about OY.

Similarly the force X_1 at A is equivalent to :
force X_1 along OX, couple zX_1 about OY and $-yX_1$ about OZ.

Also Y_1 at A is equivalent to,
force Y_1 along OY, couple $-zY_1$ about OX and xY_1 about OZ.

Hence the force P acting at A is equivalent to forces X_1, Y_1, Z_1, along OX, OY, OZ respectively, and couples

$(yZ_1, -zY_1)$, $(zX_1 - xZ_1)$, $(xY_1 - yX_1)$ about OX, OY, OZ respectively.

All other forces of the system can be treated in the same way, so that the system is equivalent to,

forces $\Sigma X_1 = X$, $\Sigma Y_1 = Y$, $\Sigma Z_1 = Z$,

and couples $\Sigma(yZ_1 - zY_1) = L$, $\Sigma(zX_1 - xZ_1) = M$, $\Sigma(xY_1 - yX_1) = N$ along and about OX, OY, OZ, respectively.

The three forces reduce to a single force R, given by

$$R^2 = X^2 + Y^2 + Z^2$$

acting through O a line whose direction cosines are X/R, Y/R, Z/R.

The three couples reduce to a single couple of moment G given by

$$G^2 = L^2 + M^2 + N^2$$

having its axis along a line whose direction cosines are L/G, M/G, N/G.

Hence the system reduces to a single force acting through a specified point O and a couple whose axis passes through O.

11·3. Conditions of equilibrium. When $R=0$ *i.e.*, $X=0$, $Y=0$, $Z=0$ and G is not zero, the resultant of the system is a couple of moment G, having its axis passing through O.

When $G=0$ *i.e.*, $L=0$, $M=0$, $N=0$ and R is not zero, the resultant is a force passing through O.

When $R=0$ as well as $G=0$ *i.e.*, X, Y, Z, L, M, N all vanish separately, there shall be neither a resultant force nor a resultant couple and the body shall be in equilibrium.

Hence the conditions of equilibrium are :—

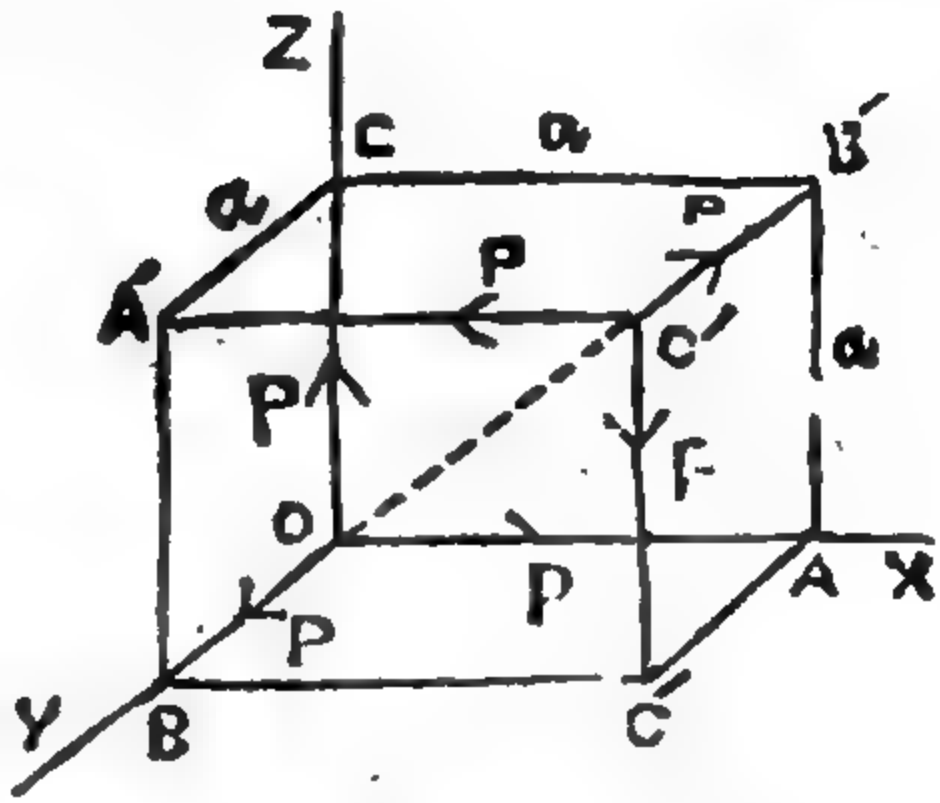
(i) *the sums of the resolved parts of the system of forces in any three directions, not all parallel to the same plane, are zero.*

(ii) *the sums of the moments of the forces about any three concurrent and non-coplanar straight lines separately vanish.*

11 . Examples. 1. *Six forces, each equal to P act along the edges of a cube which meet a given diagonal, the forces radiating from the vertices. Show that the system is in equilibrium.*

Let OO′ be the diagonal and the forces and the co-ordinate axes as shown in the figure.

Here X=P along OA
−P along O′A′=0
Similarly Y=0, Z=0.
L=sum of the moments of the forces about OX
=P along O′B′×AB′
−P along O′C′×AC′
=P×a−P×a=0, a being the length of an edge,
Similarly M=0, N=0
Hence the system of forces is in equilibrium.



2. In Ex. 1, if the forces P act in order along the edges which do not meet the diagonal OO′, show that the resultant is a couple of moment $2\sqrt{3}\,Pa$.

3. If two forces, each equal to P, act along two perpendicular diagonals of opposite faces of a cube show that they are equivalent to a single force $\sqrt{2}P$ along a line through the centre of the cube and a couple of moment $\frac{a}{\sqrt{2}}P$ with the same line for axis, a being the length of a side.

4. If the centre of the cube were fixed and the two forces in Ex. 3, act along diagonals of adjacent faces which do not meet show that the moment of the couple which will keep the cube at rest is either $\frac{1}{2}\sqrt{3}Pa$ or $\frac{1}{2}aP$ according to the direction of the forces.

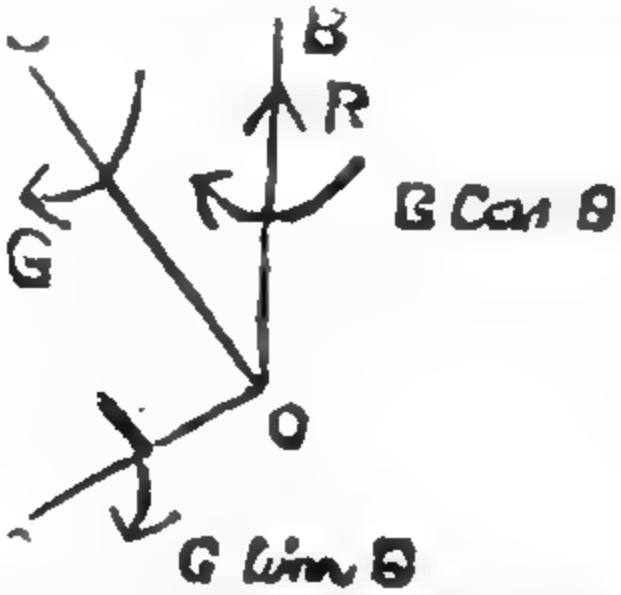
5. In the Fig. of Ex. 1, forces equal to P, 2P, 3P, 4P act along OB′, O′A, BC and C′A′ respectively. Show that they are equivalent to a force $\sqrt{35}P$ at O along a line whose direction cosines are proportional to −3, −5, 6, together with a couple $\frac{1}{2}\sqrt{114}Pa$ about a line whose direction cosines are proportional to 7, −2, 2.

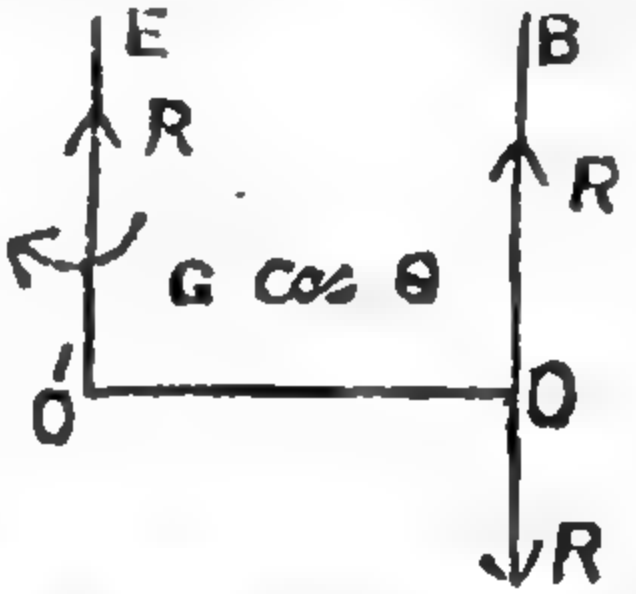
6. In the Fig. of Ex 1, equal forces each of magnitude P, act along BC′, AB′, CA′. Show that they can be reduced to a force $\sqrt{3}P$ along OO′ and a couple of moment $\sqrt{3}a\,P$ with the same line as its axis.

Show also that when these forces are reduced to the point B, the couple is of moment $\sqrt{5}a$ P with its axis parallel to the line joining O to the middle point of AC′.

11·5. The wrench. *To show that a system of forces can be reduced to a force R and a couple whose axis is the line of action of R.*

It has been shown in Art. 11·2 that any system of forces is equivalent to a force R acting at an arbitrarily chosen point O along a line, say OB (taken in the plane of the paper) and a couple of moment G whose axis passes through O.





Let OC be the axis of the couple G, making an angle θ with OB.

Now the couple G about OC as axis is equivalent to a couple G $\cos\theta$ about OB as axis and G $\sin\theta$ about OD as axis, OD being perpendicular to OB and perpendicular to the plane of the paper.

The axis of the couple G $\sin\theta$ being perpendicular to the plane of the paper, its forces act in the plane in which R also acts. This couple is now replaced by two unlike equal forces, each equal to R, one acting along BO produced and the other acting along O′E, OO′ being equal to G $\sin\theta$/R.

The two equal and opposite forces R at O balance and we are left with a force R along O′E and a couple of moment G $\cos\theta$ with axis OB.

Shifting the axis of the couple from OB to the parallel axis O′E, the system is reduced to a single force R at O′ and a couple whose axis coincides with the line of action of R.

Such a combination of a force and a couple is called a **wrench.**

O'E, the line of action of the force, is called **Poinsot's Central axis**

Taking $G\cos\theta = K$, the ratio $\frac{G\cos\theta}{R} = \frac{K}{R}$ is known as the **pitch** of the wrench.

11·6. The angle θ between the directions of R whose direction-cosines are $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$ and the axis of G, whose direction-cosines are $\frac{L}{G}, \frac{M}{G}, \frac{N}{G}$, is given by

$$\cos\theta = \frac{LX+MY+NZ}{RG} \qquad \ldots\ldots\ldots(1)$$

and the pitch p is given by, $p = \frac{G\cos\theta}{R}$

$$= \frac{LX+MY+NZ}{R^2} = \frac{LX+MY+NZ}{X^2+Y^2+Z^2} \qquad \ldots\ldots\ldots(2)$$

11·61. Condition for the resultant to be a force. If $G\cos\theta = 0$ *i.e.*, $\cos\theta = 0$ and R is not zero, the resultant of the system is a single force R acting at at O'.

By (1) Art. 11·6, $\cos\theta = 0$ if

$$LX+MY+NZ=0$$

which is the required condition

Note In this case L, M, N may not separately vanish. If they do i.e., if $L=0$, $M=0$, $N=0$, then $G=0$ and the single resultant force acts at O and not at O'.

Example. *In the figure of Ex. 1 Art. 11·4, forces each equal to P act along OA, AC', C'B and BO. Show that these four forces when compounded with any fifth force acting along CB' will reduce to a single resultant.*

Let the fifth force along CB' be Q, then in this case

$X=Q$, $Y=0$, $Z=0$,

$$L=0, \qquad M=aQ, \qquad N=aP+aP=2aP$$

$$\therefore \qquad XL+YM+ZN=0$$

Hence the resultant of the system is a single force.

11·7. Invariants. R, being the resultant of given forces moved parallel to themselves to act at O, is clearly independent of the position of the point O. This is also obvious from Art 11·5.

Hence $R^2=X^2+Y^2+Z^2$

is said to be an invariant.

Also by (1) Art, 11·6

$$LX+MY+NZ=R.G\cos\theta=RK$$

is an invariant, K being the moment of the couple.

11·8. Equations of the central axis. Let (α, β, γ) be the co-ordinates of a point on the central axis referred to co-ordinate axes through O. The system reduces to a force R at this point and the couple of moment K. If L′, M′, N′ be the components of the couple, by substituting $x-\alpha$, $y-\beta$, $z-\alpha$ for x, y, z in L, M, N of Art. 11.2, we have

$$L'=\Sigma[(y-\beta)\,Z_1-(z-\gamma)Z_1]=\Sigma(yZ_1-zY_1)-\beta\Sigma Z_1+\gamma\Sigma Y_1$$
$$=L-\beta Z+\gamma Y$$

Similarly $M'=M-\gamma X+\alpha Z$

and $N'=N-\alpha Y+\beta X.$

Now the direction cosines of R are X/R, Y/R, Z/R and those of the axis of the couple K are L′/K, M′/K, N′/K and as the line of action of the force is the same as the axis of the couple, we have

$$\frac{L'}{X}=\frac{M'}{Y}=\frac{N'}{Z}=\frac{K}{R}=\text{the pitch } p$$

Substituting the values of L′, M′, N′ and changing α, β, γ to x, y, z, we get the equations of the central axis in the form,

$$\frac{L-yZ+zY}{X}=\frac{M-zX+xZ}{Y}=\frac{N-xY+yX}{Z}=\frac{K}{R}.$$

11·9. Examples. 1. *Three forces, each equal to P act on a body at the point* (a, o, o), (o, a, o) *and* (o, o, a) *parallel to the*

rectangular axes OY, OZ and OX respectively. Find the pitch of the central axis and its equations.

Here $X=P, \quad Y=P, \quad Z=P$

$L=aP, \quad Y=aP, \quad Z=aP$

$\therefore \quad R^2=X^2+Y^2+Z^2=3P^2$

and $LX+MY+NZ=3aP^2$

$\therefore$ the pitch $=\dfrac{LX+MY+NZ}{X^2+Y^2+Z^2}=\dfrac{3aP^2}{3P^2}=a$

By Art. 11·8 the equations of the central axis are

$$\frac{aP-yP+zP}{P}=\frac{aP-zP+xP}{P}=\frac{aP-xP+yP}{P}=a$$

i.e., $x=y=z$,

a straight line through the origin equally inclined to the axes. This is otherwise obvious, the components of the resultant force and the couple being equal.

2. *Forces each equal to P act along the diagonals of adjacent faces of a cube, which do not meet. Find the pitch and the position of the wrench.*

Let the forces act along the diagonals OC′ and BC, figure Ex. 1 Art. 11·4., then

$$X=\frac{1}{\sqrt{2}}P, \qquad Y=\frac{1}{\sqrt{2}}P-\frac{1}{\sqrt{2}}P=0, \quad Z=\frac{1}{\sqrt{2}}P,$$

$$L=\frac{1}{\sqrt{2}}Pa, \qquad M=0 \qquad , \quad N=0$$

$\therefore R^2=X^2+Y^2+Z^2=\frac{1}{2}P^2+\frac{1}{2}P^2=P^2$

$$LX+MY+NZ=\frac{1}{\sqrt{2}}Pa\times\frac{1}{\sqrt{2}}P=\tfrac{1}{2}aP^2$$

$\therefore$ the pitch $=\frac{1}{2}aP^2/P^2=\frac{1}{2}a$

Equations of the central axis are

$$\frac{aP/\sqrt{2}-yP/\sqrt{2}+z\times0}{P/\sqrt{2}}=\frac{0-zP/\sqrt{2}+xP/\sqrt{2}}{0}=\frac{0-x\times0+y/\sqrt{2}}{P/\sqrt{2}}$$

or $a-y=\dfrac{x-z}{0}=y$

i.e., $x=z$ and $y=\frac{1}{2}a$.

3. *A force P acts along the axis of x, and another force nP along a generator of the cylinder $x^2+y^2=a^2$, show that the central axis lies on the cylinder*

$$n^2(nx-z)^2+(1+n^2)^2\,y^2=n^4\,a^2.$$

The force P along the axis of x may be taken to act at the origin and the force nP at any point ($a\cos\theta$, $a\sin\theta$, 0) on the given cylinder.

Thus we have,

the points	0, 0, 0	$a\cos\theta$, $a\sin\theta$, 0
forces	P, 0, 0	0, 0, nP
	$\therefore$ X=P,	Y=0, Z=nP

and $L=\Sigma(yZ-zY)=a\sin\theta.nP$, $M=-a\cos\theta.nP$, $N=0$

Equations of the central axis are :

$$\frac{a\sin\theta.nP-y.nP}{P}=\frac{-a\cos\theta.nP-zP+x.nP}{0}=\frac{yP}{nP}$$

$\therefore\ y=n^2(a\sin\theta-y)$ or $(1+n^2)y=n^2a\sin\theta$

and $nx-z=an\cos\theta$ or $n(nx-z)=n^2a\cos\theta$

Hence $(1+n^2)^2y^2+n^2(nx-z)^2=n^4a^2$.

4. Show that the following system reduces to a single force :

Force $5\sqrt{2}$ acting along the line $\frac{x}{3}=\frac{y}{4}=\frac{z-\frac{1}{5}}{-5}$

Force $3\sqrt{11}$,, ,, $\frac{x-\frac{1}{5}}{5}=\frac{y}{5}=\frac{z}{-7}$

Force $\sqrt{11}$,, ,, $\frac{x}{-3}=y-\frac{1}{4}=z$,

the co-ordinate axes being rectangular.

5. Three edges AB, AC, AD, of length a, of a cube are taken as axes of x, y and z. Forces, each equal to P, act along AD and two other edges parallel to AB and AC which intersect neither AD nor one another. Show that the central axis is the line

$$x=y+\frac{a}{3}=z-\frac{a}{3}.$$

6. Two forces act, one along the axis of y and the other along the line $y=0$, $z=-c$. As the forces vary, show that the surface generated by the axis of their equivalent wrench is $z(a^2+y^2)=cx^2$.

7. Forces P, Q, R act along the edges AB, EC, DF of a regular octahedron ABCDEF of which EF is a diagonal. Prove that they will be equivalent to a single resultant, provided

$$\frac{1}{P}+\frac{1}{Q}+\frac{1}{R}=0,$$

and that the resultant is equal to $(P^2+Q^2+R^2)^{\frac{1}{2}}$

(I.A.S.)

8. Forces X, Y, Z act along the lines
$y=b$, $z=-c$; $z=c$, $x=-a$; and $x=a$, $y=-b$ respectively ; show that they will have a single resultant if

$$\frac{a}{X}+\frac{b}{Y}+\frac{c}{Z}=0,$$

and that the equations of its line of action are any two of the three lines

$$\frac{y}{Y}-\frac{z}{Z}-\frac{a}{X}=0,\ \frac{z}{Z}-\frac{x}{X}-\frac{b}{Y}=0,\ \frac{x}{X}-\frac{y}{Y}-\frac{c}{Z}=0.$$

9. Three forces act along the straight lines
$x=0$, $y-z=a$; $y=0$, $z-x=a$; $z=0$, $x-y=a$
Show that they cannot reduce to a couple.

Prove also that if the system reduces to a single force its line of action must lie on the surface

$$x^2+y^2+z^2-2yz-2zx-2xy=a^2$$ *(P.U.)*

10. A force parallel to the axis of z acts at the point $(a, 0, 0)$ and an equal force perp. to the axis of z acts at the point $(-a, 0, 0)$. Show that the central axis of the system lies on the surface

$$z^2(x^2+y^2)=(x^2+y^2-ax)^2$$ *(P.U. 1948)*

11. Forces X, Y, Z act along the lines
$y=0$, $z=c$; $z=0$, $x=a$; $x=0$, $y=b$;

prove that pitch of the equivalent wrench is

$$(aYZ+bZX+cXY)/(X^2+Y^2+Z^2).$$

If the wrench reduces to a single force, show that the line of action of the force must lie on the hyperboloid

$$(x-a)\,(y-b)\,(z-c)-xyz=0 \qquad (M.T.)$$

12. Two equal forces act one along each of the straight lines

$$\frac{x\pm a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{\pm b\cos\theta}=\frac{z}{c};$$

Show that their central axis must, for all values of θ lie on the surface

$$y\left(\frac{x}{z}+\frac{z}{x}\right)=b\left(\frac{a}{c}+\frac{c}{a}\right)$$

13. Show that forces acting along generators of the same system of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$ are in equilibrium if they would be in equilibrium when acting at a point in the same directions.

14. Two equal forces act along generator of the same system of the hyperboloid $(x^2+y^2)/a^2-z^2/b^2=1$, and cut the plane $z=0$ at the extremities of perpendicular diameters of the circle $x^2+y^2=a^2$. Show that the pitch of the equivalent wrench is $a^2b/(a^2+2b^2)$.

CHAPTER XII

SIMPLE MACHINES

12·1. A **machine** is an apparatus in which work is done by an agent on the machine by applying a force, called the **Power** or **Effort** at one part, with the object of getting work done by the machine in overcoming some external force, called the **Weight**, or **Load** or **Resistance**, at another part.

In Mechanics the word power being also used in another sense, *viz.*, "the rate of doing work", effort is the word which is usually used. We shall denote this by the letter P, the weight or the resistance being denoted by the letter W.

12·11. In most cases the machine is so arranged that by applying a small effort P, a larger force W is overcome. If in a machine an effort P balance a resistance W, the ratio **W/P** is called the **Mechanical Advantage.**

The mechanical advantage can have any value from zero to infinity.

If x be the distance moved by the point of application of the effort P and y, the displacement of the point of application of the resistance W, in the same time, the ratio x/y is called the **Displacement Ratio** or **Velocity Ratio.**

12·12. All machines are mostly based on the combinations of simple materials such as cords, rods and planes.

When a machine is made to work, its several parts receive displacements consistent with their geometrical relations. The work done by the effort P is equal to that of the resistance W together with the work of the resistance of the machine such as friction etc. Thus the work done on W, called the *useful work* of a machine is usually less than that actually done by P.

If, however, all parts of a machine are quite smooth and the connections are all weightless, then,

$$Px = Wy \text{ i.e., } W/P = x/y$$

so that *the mechanical advantage is equal to the velocity ratio.*

If W>P then $x>y$, so that the effort P has to undergo a greater displacement. This is sometimes expressed by saying "What is gained in power is lost in speed."

12·13. The **efficiency** of a machine is the ratio of the useful work to that done by the power, so that

$$\text{The efficiency} = \frac{\text{the work output}}{\text{the work input}} = \frac{Wy}{Px}$$

$$= \frac{W}{P} \div \frac{x}{y} = \frac{\text{mechanical advantage}}{\text{velocity ratio}}.$$

It shows that the efficiency of a machine would be unity if all its parts were perfectly smooth, weightless and rigid.

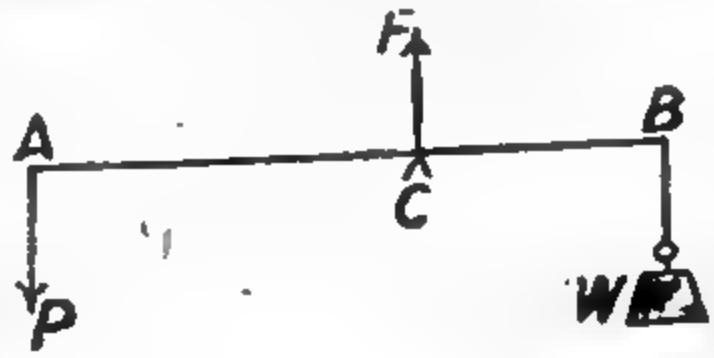
In all existing machines, however, where, these ideal conditions do not exist, Wy is always less than Px and the efficiency is necessarily less than unity.

We now give a brief account of some of the principle simple machines such as the lever, the pulley, the wheel and axle, the inclined plane, the balance and the screw.

12·2. The Lever. A lever is a rigid rod, straight or bent, movable about a fixed axis called the *fulcrum*. The portions of the lever between the fulcrum and the points of application of the effort P and the weight W are called *arms* of the lever, P and W being supposed to act in a plane perpendicular to the axis of the lever.

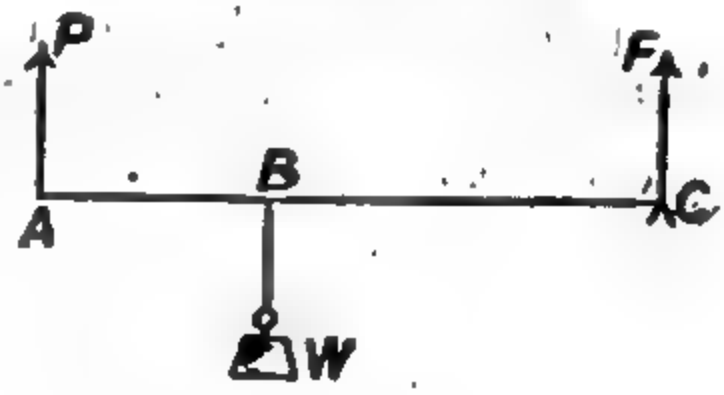
In its simplest form a lever is a straight rigid rod which has one fixed point about which the rod turns. There are three kinds of levers the classification depending on the position of the fulcrum relative to the lines of action of the forces.

12·21. Lever Class I. In this case P and W are on opposite sides of the fulcrum C, F being the reaction at the fulcrum, and AC>BC.



A common balance, a pair of scissors are examples of this kind of lever.

12·12. Lever Class II. Here P and W act on the same side of the fulcrum C, W being between P and F.



A nut-cracker ; an oar, the fulcrum being the end in water ; a wheelbarrow ; a crowbar, the end on the ground being the fulcrum, are levers of this type.

12·23. Lever Class III. Here P and W are on the same side of C but P acts between W and F.

A pair of sugar-tongs ; the human forearm, the weight being carried on the palm of the hand, the fulcrum being at the elbow, are levers of this type.

Neglecting the weight of the lever and taking moments about C in each case, we have

$$P.AC = W.BC \quad i.e., \quad W/P = AC/BC.$$

Thus there is a mechanical advantage in classes I and II but in class III, there is a mechanical disadvantage, P being greater than W.

EXAMPLES XXXVII

1. A man who wishes to raise a rock leans with his whole weight on one end of a crowbar 5 feet long which is propped at the distance of 4 inches from the end in contact with the rock. The man's weight being 160 lb , what force does he exert on the rock, and what is the pressure on the prop ?

[*Ans.* 2240,2400 lb.]

2. Two forces 6 and 8 lb. act at the ends A and B of a rod 16 ft. long and are inclined to the rod at angles of 30° and 60° respectively ; find the position of the fulcrum when there is equilibrium.

[*Ans.* Nearly 11·2 ft. from A]

3. An oar is 9 ft. long and the button is 3 ft. from the handle. What is the propelling force on a boat when an effort of 20 lb. is exerted by the oarsman ?

[*Ans.* 30 lb.]

4. A man seated in a boat pulls at the handle of each

of a pair of sculls with a force of 25 lb. wt. If the distance of the rowlock from the end of the blade of each scull be 4 times that of the rowlock from the hand, find the resultant force on the boat. [*Ans.* $62\frac{1}{2}$ lb.]

5. A pair of nut-crackers are 6 inches long. A pressure of 10 lb. is required to crack a nut which is pressed at a point 3/4 in. from the hinge. What weight placed at the top of the nut will crack it ? [*Ans.* 80 lb.]

6. The arms of a light bent lever are at right angles to one another and are in the ratio of 5 to 1. The longer arm is inclined to the horizon at an angle of 45°, and carries at its end a weight of 10 lb. ; the end of the shorter arm passes against a horizontal plane, find the pressure on the plane. (*Ans.* 50 lb. wt.)

7. The arms of a light bent lever measure 8 and 24 inches; and they include an angle of 75°. If a mass of $9\sqrt{2}$ lb. is attached to the end of the shorter arms and another of 3 lb. to the end of the longer arm, show that in the position of equilibrium, the shorter arm makes an angle of 60° with the horizontal.

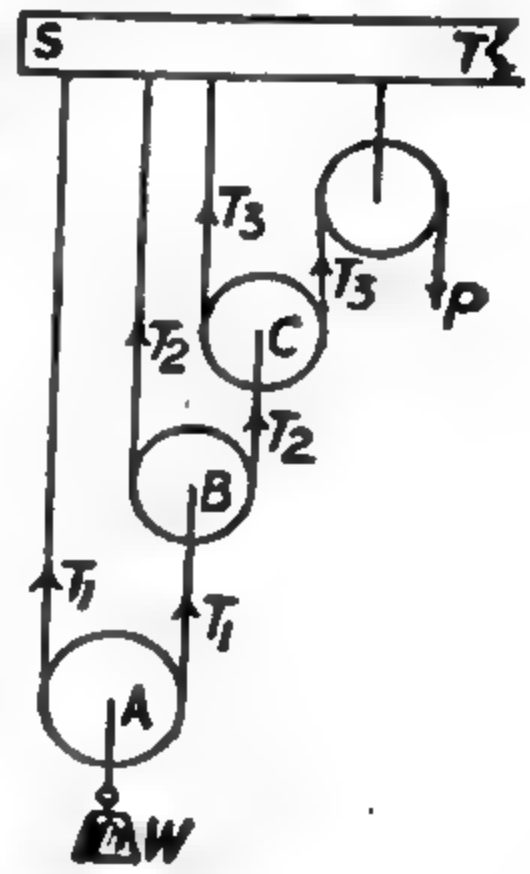
8. Two weights P and Q are supported from the extremities of a lever without weight, in the form of an arc of a circle subtending an angle 2α at the centre, which rests with its covexity downwards upon a horizontal plane. Show that the angle θ which the central radius makes with the vertical in the position of equilibrium is given by

$$(P+Q)\tan\theta=(P-Q)\tan\alpha.$$

12·3. The pulley. A pulley consists of a wheel grooved along its circumference to receive a rope. It can turn freely about an axis through its centre perpendicular to its plane.

Usually more than one pulleys are arranged to form a system. For lifting heavy weights three systems of pulleys are commonly used, which we now deal with separately.

12.31. First System of Pulleys. *Each string attached to a suppoting beam.* This consists of a number of movable pulleys (only 3 taken in the figure) so arranged that the string passing round each has one end attached to the supporting beam ST, and the other end to the next higher pulley, the weight W, being attached to the lowest pulley.



The highest *fixed* pulley is used simply for applying the effort P in the convenient downward direction and does not otherwise affects the mechanical advantage.

The parts of the strings not in contact with the pulleys are assumed to be all vertical and friction is neglected.

Let w_1, w_2, w_3 be the weights of the pulleys, A,B,C and let T_1, T_2, T_3 be the tensions as shown in the diagram.

Now for equilibrium, we have

$$2T_1 = W + w_1, \quad 2T_2 = T_1 + w_2 = \frac{W}{2} + \frac{w_1}{2} + w_2$$

$$2T_3 = T_2 + w_3 = \frac{W}{2^2} + \frac{w_1}{2^2} + \frac{w_2}{2} + w_3$$

$$\therefore \quad P = T_3 = \frac{W}{2^3} + \frac{w_1}{2^3} + \frac{w_2}{2^2} + \frac{w_3}{2}$$

so that for n movable pulleys, we must have

$$P = \frac{W}{2^n} + \frac{w_1}{2} + \frac{w_2}{2^{n-1}} + \frac{w_3}{2^{n-2}} + \ldots\ldots\ldots + \frac{w_n}{2}$$

$$\text{or } 2^n P = W + w_1 + 2w_2 + 2^2 w_3 + \ldots\ldots + 2^{n-1} w_n \qquad \ldots\ldots(1)$$

If the pulleys be equally heavy each of weight w, then

$$w_1 = w_2 = w_3 = \ldots\ldots = w_n = w$$

$$\therefore \quad 2^r P = W + w(1 + 2 + 2^2 + \ldots\ldots + 2^{n-1}) = W + w(2^n - 1) \qquad \ldots\ldots(2)$$

If the pulleys are of negligible weights, then $w = 0$.

$$\therefore \quad 2^n P = W \qquad \text{i.e.} \quad W/P = 2^n.$$

It follows that the mechanical advantage W/P depends on the weights of the pulleys, it being greater when the pulleys are lighter.

Cor. *The pressure on the supporting beam, when P is acting downwards, the string to which it is applied passing over a fixed pulley.*

$$=W+w_1+w_2+\ldots\ldots+w_n+P.$$

12·311. The effort can also be obtained from the principle of Virtual Work.

Suppose W gets a small upward displacement x, then the pulley A rises by x, and since due to this rise, a length $2x$ of the string slips round A the pulley B rises by amount $2x$. Similarly the pulley C will rise by a distance $4x$ and so on. If we confine ourselves to the case of three movable pulleys as shown in the diagram, the power P will consequently have a displacement equal to $8x$. Thus by the principle of work, we have

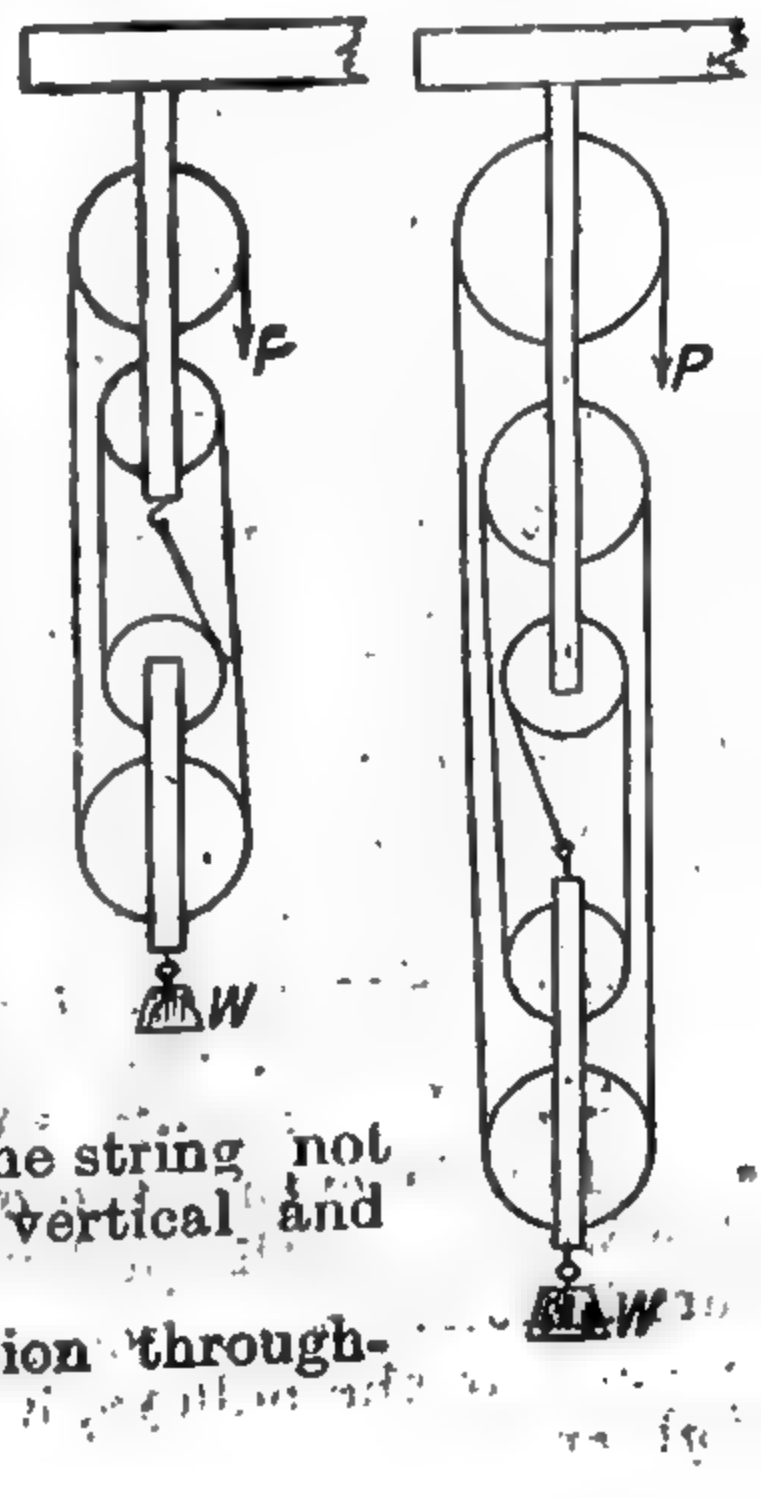
$$P.8x=W.x+w_1.x+w_2\,2x+w_3.4x$$

or
$$2^3.P=W+w_1+2w_2+2^2w_3$$

which is the same as (1) for $n=3$.

12 32. Second system of Pulleys. *The same string passing round the pulleys.*

In this system there are two blocks each containing pulleys. The upper block is fixed to a supporting beam and the lower of weight w, to which the load W is attached, is movable.



The number of pulleys in the upper fixed block is either the same or is greater by one than the number in the lower block. In the first case the end of the string is fastened to the upper block, so that the number of portions of the string connecting the blocks is *even*. In the second case the end of the string is fastened to the lower block, the parts of the string connecting the blocks being *odd*.

It is assumed that the parts of the string not in contact with the pulleys are all vertical and that the pulleys are smooth.

The pulleys being smooth, the tension through-

out the string is the same, equal to effort P. If there are n portions of string supporting the lower block, the total upward force on this block is nP.

$$\therefore \quad nP = W + w.$$

12.321. The same result can also be obtained *by the principle of the Virtual Work.*

Let the load W and the lower block rise a small distance x. It is clear that each of the n parts of the string is slackened by x. To tighten the string, P must, therefore, descend a distance x for each separate portion of the string, so that P must descend a distance nx.

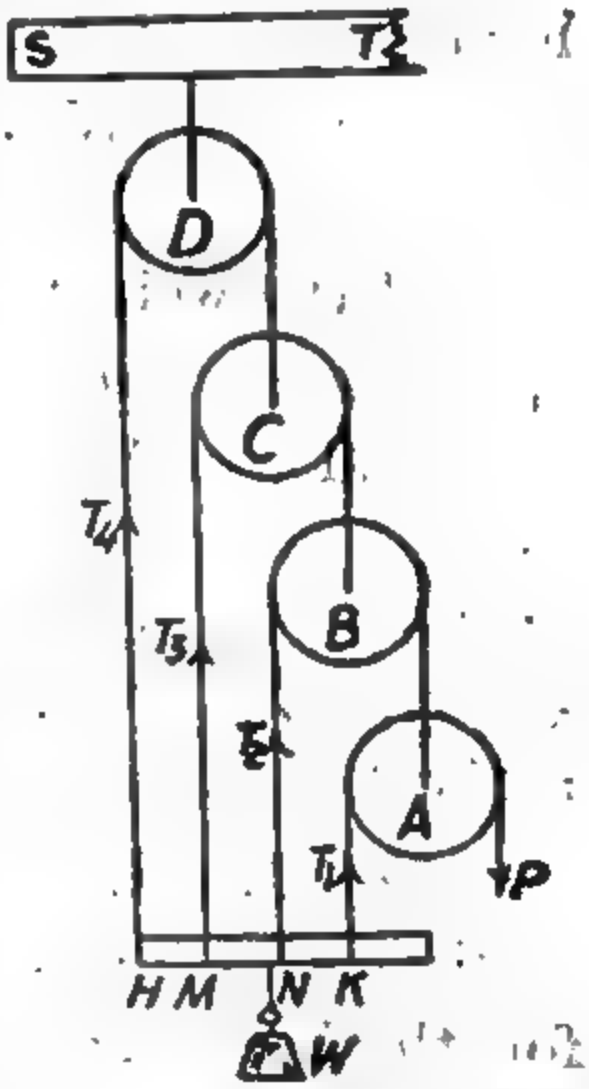
$\therefore$ by the principle of work, we have

$$P.nx = (W + w)x \qquad i.e., \; nP = W + w.$$

Note. Actually the pulleys in each block are placed beside one another, being free to rotate on the same axle.

12.33. Third System of Pulleys. *Each string attached to the weight.*

In this system one end of each string passing over any pulley is fastened to a bar HK supporting the load W, the other end being attached to the next lower pulley, the highest pulley being fixed to the supporting beam ST. The effort P is applied to the free end of the string passing over the lowest pulley. This system is, in fact, the inverted first system.



It is assumed that the free portions of the strings are all vertical and that the pulleys are all smooth.

Let T_1, T_2, T_3...be the tensions of the strings passing over the pulleys A, B, C.....

I. Let us first neglect the weights of the movable pulley, then we have

$$T_1 = P, \qquad T_2 = 2T_1 = 2P$$
$$T_3 = 2T_2 = 2^2P, \qquad T_4 = 2T_3 = 2^3P$$

$$\therefore \quad W=T_1+T_2+T_3+T_4=(1+2+2^2+2^3)\,P=(2^4-1)\,P$$

if there were n pulleys, of which $(n-1)$ are movable, we should have

$$W=(2^n-1)\,P \qquad \ldots\ldots(1)$$

Thus the mechanical advantage is 2^n-1.

To keep the bar HK horizontal, the load W has to be attached at a proper place, which can be found by taking moments about some point, say H.

II. Now let us suppose that the movable pulleys A, B, C have weights w_1, w_2, w_3, then

$$T_1=P$$
$$T_2=2T_1+w_1=2P+w_1$$
$$T_3=2T_2+w_2=2^2P+2w_1+w_2$$
$$T_4=2T_3+w_3=2^3P+2^2w_1+2w_2+w_3$$
$$\therefore \quad W=T_4+T_3+T_2+T_1$$
$$=(2^3+2^2+2+1)\,P+(2^2+2+1)w_1+(2+1)w_2+w_3$$
$$=(2^4-1)\,P+(2^3-1)w_1+(2^2-1)w_2+(2-1)w_3.$$

If there are n pulleys, of which $(n-1)$ are movable, we have

$$W=(2^n-1)P+(2^{n-1}-1)w_1+(2^{n-2}-1)w_2+\ldots\ldots\ldots$$
$$\ldots\ldots+(2^2-1)w_{n-2}+(2-1)w_{n-1}.$$

If the weight of each pulley is the same, w, then

$$w_1=w_2=\ldots\ldots=w_{n-1}=w$$

and
$$W=(2^n-1)P+w[2^{n-1}+2^{n-2}+\ldots+2-(n-1)]$$
$$=(2^n-1)P+w(2^n-1-n)=(2^n-1)(P+w)-nw.$$

The mechanical advantage is increased by increasing the weights of the pulleys. In this case the weights of the pulleys assist the power in sustaining the load.

12 331. The **pressure** en the supporting beam ST

$$=W+w_1+w_2+\ldots\ldots w_n+P$$

for, the beam has to support the load W, the weights of all the pulleys, including the fixed one, and the effort P.

This is also equal to the sum of the weight of the fixed pulley and twice the tension of the string passing over it.

EXAMPLES XXXVIII

1. In the first system, there are 4 pulleys, of equal weight each weighing 2 lb. What weight can be raised by a force equal to the weight of 20 lb. ?

What is the weight of each pulley if a power of 4 lb. supports a weight of 49 lb. [*Ans.* 290 lb. ; 1 lb.]

2. In the first system of 4 pulleys of equal weight a certain weight can be supported by a force of 7 lb. wt., but if a fifth similar pulley be used the same weight can be supported by a force of 4 lb. wt. Find the supported weight and the weight of each pulley. [*Ans.* 97 lb. ; 1 lb.]

3. Three single pulleys are arranged as in the first system. What effort will be required to raise a load of 40 lb. if

(1) the pulleys are light but due to friction the efficiency is 80 per cent ?

(2) the weight of each pulley is 2 lb. and there is no friction. [*Ans.* 6·25 ; 6·75 lb.]

4. A man weighing 13 stone suspended from the lowest of a system of 6 light pulleys of the first system, supports himself by pulling at the end of the string that passes over the topmost fixed pulley Show that he must have applied a muscular effort of $2\frac{4}{8}$ lb. wt.

5. A man weighing 10 stone supports a weight of 91 lb. by means of three movable pulleys arranged in the system where each pulley hangs in the loop of a separate string. The pulleys weigh 5 lb., 4 lb. and 2 lb. respectively. What is the thrust of the man on the ground ? [*Ans.* 9 stone]

6. There are four movable pulleys in the first system, each of weight w and the effort acting upwards (the string to which it is applied not passing over a fixed pulley) is P. Show that the stress on the beam is $15P - 11w$.

7. In the second system of pulleys, if the lower block weights 24 lb. and contains 5 pulleys, the string being fastened to the lower block, what weight can be raised by a force equal to 21 lb. wt. [*Ans.* 207 lb.]

8. If weights of 5 lb. and 6 lb. support weights of 73 lb.

and 88 lb. respectively, what is the weight of the lower block. and how many pulleys are there in it ? [*Ans.* 2 lb. : 7]

9. We are told that the cable by which "Great Paul" whose weight is 18 tons, was lifted into its place in the Cathedral tower, passed four times through the two blocks of pulleys. From this statement give a description of the pulleys and estimate the strength of the cable. (*Roorkee*)

[*Ans.* Can support more than $2\frac{1}{4}$ tons.]

10. A man weighing 12 stone supports a weight of 3 cwt. by means of a system of pulleys, there being four pulleys in each block and the string being attached to the upper block. What is his pressure on the ground ?

[*Ans.* 9 stone wt.]

11. Five pulleys are arranged in two blocks—the upper fixed block having three, and lower movable block two.

A man weighing 120 lb. sits on a board attached to the lower block and pulls himself up by means of a rope passing round all the pulleys. If the lower block and board weigh 10 lb., find the smallest force that he must exert to raise himself.

If the free end of the rope descends 10 ft., how much useful work is done ? [*Ans.* $21\frac{2}{3}$ lb.; 260 ft. lb.]

12. Find the least number of pulleys in a movable block weighing 10 lb. in order that a weight of 120 lb. may be lifted by a downward force not exceeding 28 lb. Find also the necessary downward force. [*Ans.* 2 ; 26 lb.]

13. A pulley system in which there is only one string, consists of four pulleys. When on the point of motion the tension of the string as it passes over each pulley is increased 25 per cent. Find the force which will just lift a weight of 246 lb. [*Ans.* 104·2 lb. wt.]

14. If four pulleys (three movable) each weighing 2 lb. be used in the third system, find the force required to support a weight of 247 lb. [*Ans.* 15 lb wt.]

15. By means of a third system of pulleys with 6 light pulleys, a man weighing 12 stone supports a weight 18 stone. Find the pressure of the man on the ground. [*Ans.* 11 st. 10 lb.]

16. If there are n light pulleys in the third system, each pulley of diameter $2a$, show that the distance of the point of application of the weight from the line of action of the effort is $2^n.na/(2^n-1)$.

17. If there are three movable pulleys, each of weight w and radius a, show that for the bar carrying the weight to remain horizontal, the distance of the C. G. of the bar and the weight, from the point of attachment of the longest string passing over the fixed equal pulley must be equal to $(11P+5w)a/W$, where W is the weight of the bar including the attached weight.

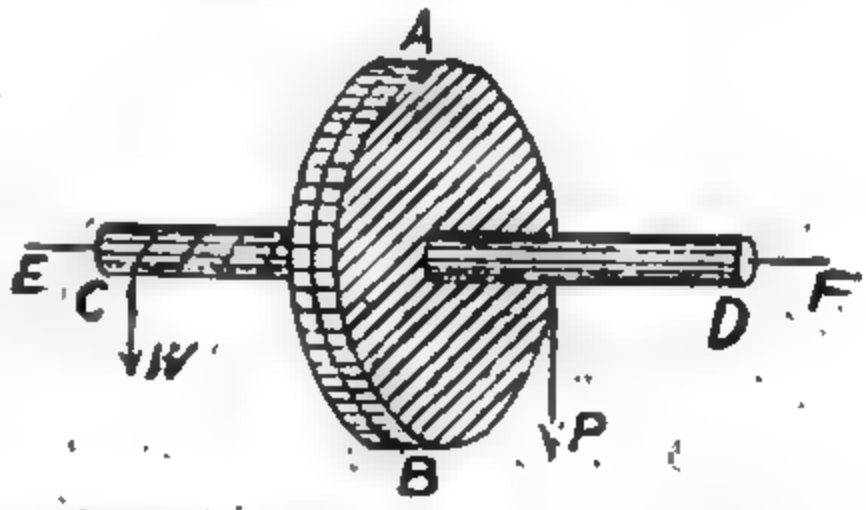
18. In a system of n equal weightless pulleys, in which the string passing round any pulley has an end attached to a weightless bar AB, there are $(n-1)$ movable pulleys and one fixed pulley; the string round the fixed pulley is attached to the bar at A, and that round the last movable pulley is attached to the bar at B and has a weight P at the other extremity. The bar AB is kept in horizontal equilibrium by a weight W attached at A and another weight at B; prove that

$$W=[(n-2)\ 2^n+2]\ P/(n-1).$$

12·4 The Inclined Plane. The equilibrium of a weight on a smooth or rough inclined plane under the action of an effort P acting in different directions has already been dealt with in II and III Chapters.

12·5. The Wheel and Axle. *To find the relation between the effort and the weight.*

This machine consists of a *wheel* AB rigidly connected to a horizontal cylinder, the *axle* CD, the plane of the wheel being perpendicular to the axle. The whole thing turns freely about the common axis EF.



The effort P is applied to a string wound round the wheel and the weight W, to be lifted, is attached to another string wound round the axle in the opposite direction.

If a be the radius of the wheel and b that of the axle, by taking moments about the axis, we have

$$P\,a = W.b \qquad i.e., \qquad W/P = a/b,$$

The same result is also given by the principle of Virtual Work.

Let $\delta\theta$ be the small angle through which the machine turns. When P descends a distance $a\delta\theta$, W moves up a distance $b\delta\theta$.

Hence by the principle of work, we have

$$W.b\delta\theta = P.a\delta\theta \qquad i.e., \qquad W/P = a/b$$

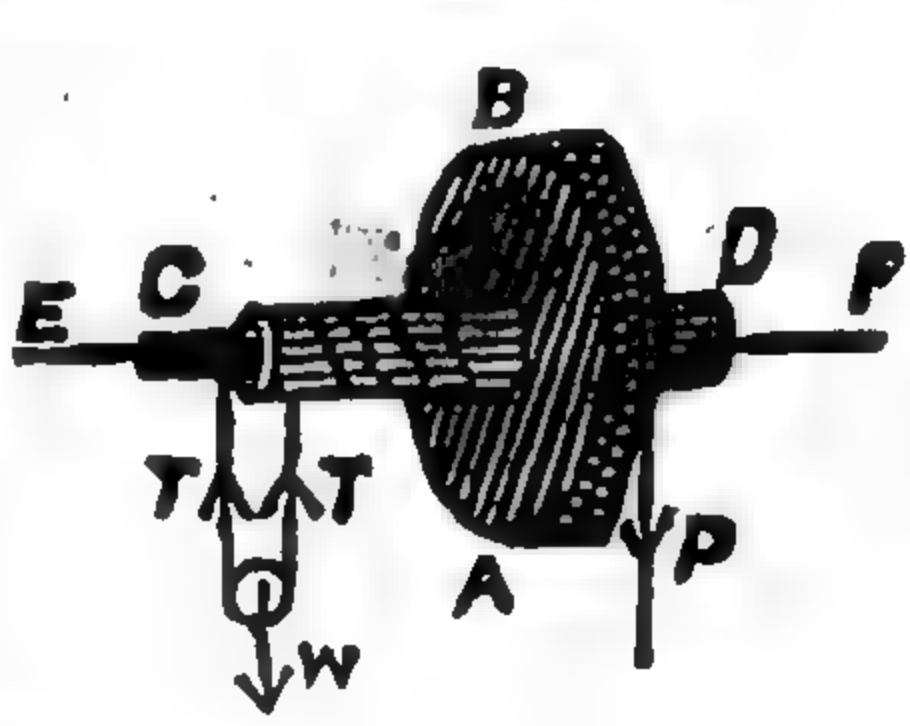
Theoretically, the mechanical advantage can be increased by making a large or b small but in practice if a is very large, the machine becomes unwieldy, and if b be small, the axle may break.

A Windlass and a Capstan are only modified forms of the wheel and axle In these the effort is applied at the ends of spokes or long handles perpendicular to the axle.

In a Windlass, which is used for drawing water, the axle is horizontal. In a Capstan, used on board the ship, the axle is vertical.

12·51. The Differential Wheel and Axle.

With a view to get greater mechanical advantage without making the machine unwieldy, the wheel and axle is modified into the differentlal wheel and axle in which the axle consists of two parts of radii, say b and c $(b>c)$. The weight W is attached to a pulley and the rope supporting this is wound in opposite directions on the two parts of the axle. As P descends, the rope is wound up on the larger axle and unwinds on the smaller.



For the equilibrium of the weight, we have

$$2T = W \qquad \text{or} \qquad T = \tfrac{1}{2}W.$$

Taking moments about the central axis EF, we have

$$P.a+T.c=T.b$$

$$\therefore\ P=T(b-c)/a=W(b-c)/2a \qquad \therefore\ W/P=2a/(b-c)$$

So that the mechanical advantage can be increased by making the difference $b-c$ small, without making the wheel too large or the axle too thin.

EXAMPLES XXXIX

1. The handle of a winch has a radius of 18 in., and the drum is 3 in. in diameter. Find the force on the handle necessary to give a pull of 6 cwt. on the rope.

[*Ans.* 56 lb. wt.]

2. A cage is suspended by a rope over an axle and a man standing in the cage draws himself and the cage up by pulling at a rope passing over the circumference of the wheel. If the joint weight of the man and cage be 15 stone and the radius of the wheel six times that of the axle, find the tension exerted by the man.

If the rope is being pulled at the rate of 5 ft. a second, find the horse-power at which the man works.

[*Ans.* 30 lb. wt. ; 3/11 H. P.]

3. A man pushing at the end of a pole 5 ft. long works a capstan, whose diameter is 2 ft. ; with what force must he push to overcome a resistance of 600 lb. wt. ?

[*Ans.* 100 lb. wt.]

4. Four sailors each exerting a force capable of raising 119 lb. raise an anchor by means of a capstan whose radius is 2 ft. 4 in. and whose spokes are 8 ft. long (measured from the axis). Find the weight of the anchor. [*Ans.* 1632 lb.]

5. In a differential wheel and axle, if the radius of the wheel be one foot and the radii of the two portions of the axle 4 and 2 inches, what force will support a weight of 36 lb. ?

[*Ans.* 3 lb. wt.]

6. In a differential wheel and axle the two parts of the axle are of radii 4 and 5 inches, and the force is applied at the end of an arm 18 inches long. To raise a weight of 3000 lb. a force of 96 lb. is required. What is the efficiency ? [*Ans.* 86·8]

7. In a differential wheel and axle the two parts of the rope on which the weight W is slung make an angle 2α with one another, and a, b, c are the radii of the two parts of the axle and of the wheel, prove that

$$Pc = \tfrac{1}{2}W(b-a)\sec\alpha.$$

8. In a wheel and axle the radius of the wheel is a and that of the axle is b, the weight supported is W and the rope at which the effort is applied is perpendicular to the axis. making an angle θ to the vertical. Show that the magnitude of the thrust on the bearings of the machine is

$$W\sqrt{a^2+b^2+2ab\cos\theta}/a.$$

9. The shaft of a fly-wheel rests horizontally in two V grooves, the angle of each being 60°. A rope round the wheel supports a weight P. The co-efficient of friction is μ and the wheel and the shaft weigh W. Show that the least value of P which will cause the wheel to rotate in the grooves is given by

$$P = 2\mu rW/[(1+\mu^2)R - 2\mu r],$$

where r is the radius of the shaft and R that of the wheel.

(roorkee)

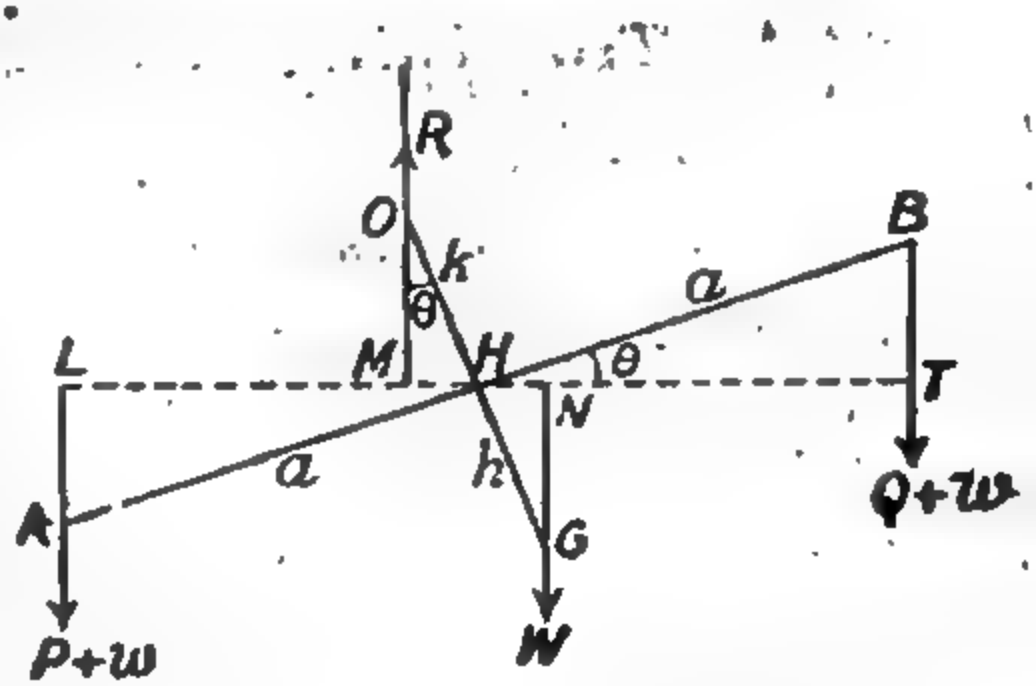
126. The Common Balance. This consists essentially of a rigid beam AB carrying scale-pans at its extremities. The beam with the scale pans turn freely about the fulcrum O, outside the beam but rigidly connected with it by a rod which bisects the beam at right angles at H. G, the C. G. of the beam together with the scale pans lies on the line OH, a little below the beam. (See Fig. P. 285.)

A balance is *true* when the arms AH, BH and the weights of the scale pans are equal and O, H, G are in a straight line perpendicular to the beam AB.

If these conditions are satisfied, the beam remains horizontal when the scale pans are empty or when equal weights are placed in them. For unequal given weights in the scale pans, the beam will rest in a definite position inclined to the horizontal.

12·61. *To find the relation between the inclination of the beam and the weights in the pans.*

Let AB be the beam, O the fulcrum, and G, the C.G. of the beam.



Let w be the weight of each scale pan, P and Q the weights placed in them, W the weight of the beam and θ its inclination to the horizontal.

Let $AH=HB=a$, $OH=k$ and $HG=h$. Draw LHT, the horizontal line through H.

Taking moments about the fulcrum O, we have

$$(P+w)\times LM=W\times MN+(Q+w)MT$$

or $(P+w)(LH-MH)=W(MH+HN)+(Q+w)(MH+HT)$

i.e., $(P+w)(a\cos\theta-k\sin\theta)=W(k\sin\theta+h\sin\theta)+(Q+w)(k\sin\theta+a\cos\theta)$

$$[(P+w)a-(Q+w)a]\cos\theta=[(P+w)k+(Q+w)k+W(h+k)]\sin\theta$$

$$\tan\theta=\frac{(P-Q)a}{(P+Q+2w)k+W(h+k)}.$$

12·62. This result shows that :—

(1) If $P=Q$, $\theta=0$, so that if the weights in the pans are equal, the beam can only rest in a horizontal position.

(2) If the weights P and Q are unequal, the beam will rest inclined at a definite angle to the horizontal.

(3) If $h=k=0$, $\theta=\frac{1}{2}\pi$ and if $h=k=0$ and also $P=Q$, θ may have any value ; so that if the centre of mass of the beam and the centre of suspension coincide in the line AB, the beam can rest in any position when equal weights are placed in the pans and can rest only in a vertical position if the weigths are unequal.

This can be avoided by having G below AB.

12·63. *The requisites of a good balance are that it must be* :—

(i) *true* (ii) *sensitive* (iii) *stable* (iv) *rigid.*

12.631. True balance requires that (*a*) its arms should be equal (*b*) the waights of the scale pans should be equal and (*c*) the C. G. of its beam should lie in a line through the fulcrum perpendicular to the beam.

Double Weighing. If a balance is *false* due to the scale pans being of different weights or the arms of the beam being unequal, tne correct weight of a body can be obtained as follows :—

Place the body in one scale pan and balance it by placing sand in the other pan. Now remove the body and put weights in the pan until they balance the sand.

These weights must be equal to the weight of the body.

This is known as *Borda's method.*

12.632. Sensitiveness of a balance requires that for a small difference between tne weights in the pans, the beam must be inclined at an appreciable angle to the horizontal.

For a given difference P—Q of the weights tan $\theta/(P-Q)$ may be taken as the measure of the Sensitiveness. But this, by the result of Art. 12.61 is equal to

$$\frac{a}{(P+Q+2w)k+W(h+k)}$$

so that the sensitiveness of a balance will be greater if

(*a*) the arm, a, is fairly long.

(*b*) as compared to h and k, W and w the weights of the beam and the pans are as small as possible.

Thus it is always desirable to have a long and light beam.

(*c*) If k is not zero, the sensitiveness diminishes as the weights P and Q increase. If, however, $k=0$, *i.e.*, the fulcrum lies in the beam,

$$\text{Sensitiveness} = \frac{a}{Wh}$$

which is independent of the weights in the pans.

This condition is generally aimed at and k is made as small as possible consistent with the rigidity of the balance.

12.633. Stability of a balance requires that it must quickly take up its position of equilibrium, *i.e.*, the time of an oscillation should be small. From a Dynamical consideration it is found that this is possible when a is small and h and k large. These conditions are, therefore, the reverse of those required for sensitiveness. Thus it shows that sensitiveness and quickness in weighing both cannot be attained simultaneously.

EXAMPLES XL

1. *The arms of a balance are of unequal lengths, but the beam remains in a horizontal position when the scale-pans are not loaded; if a body is placed in each scale-pan in succession and weighed show that its true weight is the geometric mean between its apparent weights.*

Show also that if a tradesman weighs equal quantities of a substance, using alternately each of the scale-pans, he will be a loser.

Since the beam remains horizontal when unloaded, the C. G. of the beam and the scale pans is vertically below the fulcrum.

Let a and b be the lengths of the arms of the beam, and let W_1, W_2 be the apparent weights in the two scale pans of a body whose real weight is W, then

$$W.a = W_1.b$$

and $$W.b = W_2.a$$

$$\therefore \quad W^2 = W_1 W_2 \quad \text{or } W = \sqrt{W_1 W_2}.$$

Again, by weighing apparently equal quantities in the two scale pans the tradesman really gives the customers $W_1 + W_2$ instead of $2W$.

Now $W_1 + W_2 - 2W = W_1 + W_2 - 2\sqrt{W_1 W_2} = (\sqrt{W_1} - \sqrt{W_2})^2$ a positive quantity, so that $W_1 + W_2 > 2W$, and hence the tradesman is a loser.

2. *The arms of a balance are equal in length but the beam is unjustly loaded; if a body be weighed in each scale pan successively, show that its true weight is the arithmetic mean between its apparent weights.*

Let the length of each arm be a, the horizontal distance of the C. G. of the beam from the fulcrum be x and let w be the weight of the beam, then if W_1, W_2 be the apparent weights of a body whose real weight is W, we have

$$W.a = W_1.a + w.x$$

and $$W_2.a = W.a + w.x$$

$$\therefore \quad (W - W_2)a = (W_1 - W)a \quad i.e., \; W = \tfrac{1}{2}(W_1 + W_2)$$

3. The apparent weights of a body are 4 and 16 lb. when weighed from two pans of a balance. Find the ratio of the lengths of the arms and the true weight of the body.

[*Ans.* 2 : 1 ; 8 lb.

4. If the arms of a false balance be without weight, and one arm longer than the other by $\frac{1}{9}$ of the shorter arm, and if in using it the substance to be weighed is put as often in the one scale pan as in the other, show that the seller loses $\frac{5}{9}$ per cent in his transaction.

5. The beam of a tradesman's balance is horizontal when both scale pans are unloaded, but the arms are of unequal lengths a and b. Show that if the tradesman weighs out quantities of the same substance which appear to be equal, using alternately each of the scale pans, then he loses on his transactions and that his actual receipts ought to amount to $\frac{50(b-a)^2}{ab}$ per cent more.

6. The arms of a false balance are a and b and weight W balances P at the end of the shorter arm b, and Q at the end of the arm a : show that

$$a : b = (P - W) : (W - Q).$$

7. A uniform heavy rod AB can turn freely about a fulcrum at a point whose distances from the ends are a and b. If a body is hung from A it balances a weight W hung from B, and if hung from B it balances a weight W′ hung from A. Show that the weight of the body is

$$(bW + aW')/(a + b).$$

8. In a false balance, a weight P appears to weigh Q and a weight P′ to weigh Q′ ; prove that the real weight X of what appears to weigh Y is given by

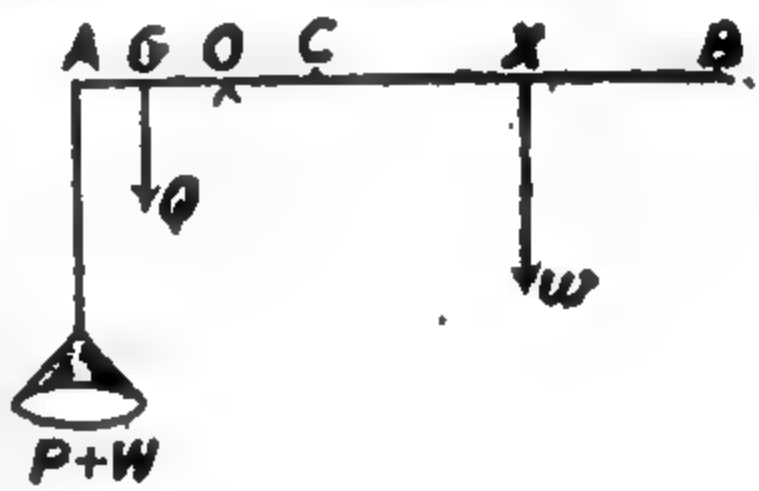
$$X(Q - Q') = Y(P - P') + P'Q - PQ'.$$

9. If a balance be unjustly loaded, and have unequal arms, and if a tradesman weigh out a customer a quantity $2W$ of some substance by weighing equal portions in two scale pans, show that he will defraud himself if the C. G. of the beam be in the longer arm.

10. It is observed that a body weighs W_1 when suspended at one end of a false balance of negligible weight and of length $2a$, but weighs W_2 when suspended at the other end; prove that the fulcrum should be shifted towards the first end through a distance

$$(\sqrt{W_1}-\sqrt{W_2})a/(\sqrt{W_1}+\sqrt{W_2}).$$

127. The Common Steelyard. This is a machine for weighing bodies. It consists of a horizontal rod AB movable about a fixed fulcrum O, nearer to the one end than the other.



At A, the end of the shorter arm, is attached a scale pan or a hook to carry the body whose weight is required, and on the longer arm OB slides a movable weight w.

Let P be the weight of the scale pan, Q the weight of the rod acting at G.

Let C be the position at which w must be placed to keep the rod horizontal when there is no weight in the plan.

Let $OA=a$, $OG=b$ and $OC=c$.

Taking moments about O, we have

$$wc=P.a+Q.b \qquad \ldots\ldots(1)$$

Now when a weight W is placed in the pan, let x be the distance from C at which the weight w will balance, then

$$w(c+x)=(P+W)a+Qb \qquad \ldots\ldots(2)$$

Subtracting (1) from (2) we get

$$wx=Wa \qquad \ldots\ldots(3)$$

It shows that if $x=a$, $W=w$, if $x=2a$, $W=2w$ and so on.

Thus if from C towards B, we mark off points at distances a, $2a$, $3a$, etc., the weight in the scale pan which will balance w at the first mark is equal to w, the one that balances w at the second mark will be $2w$ and so on.

Now if w is taken 1 lb, the graduations will correspond to pounds in the pan. These graduations can further be subdivided to read ounces.

An instrument of this form was used by the Romans and is therefore often called the Roman Steelyard.

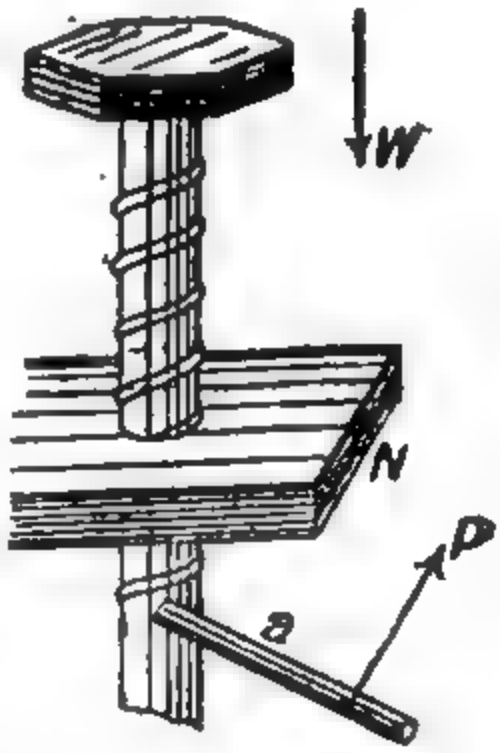
Ex. 1. A uniform rod 2ft. long and weighing 3lb. is to be used as a steelyard. The fulcrum is 2 inches from one end of the rod and the sliding weight is 1 lb. Find the greatest weight which can be determined by this machine, and also where the sliding weight must be placed to indicate a weight of 20 lb. [*Ans.* 26 lb. ; at mid-point] (*P. U.*)

Ex. 2. The weight of a steelyard is 8 lb., the body to be weighed is suspended from a point 4 inches from the fulcrum and the C. G. of the steelyard is 3 inches on the other side of the fulcrum. Where should the graduation corresponding to one cwt. be placed, the movable weight being 12 lb. ?

[*Ans.* $35\frac{1}{3}$ in. from fulcrum.

12·8. The Screw. *To find the relation between the effort and the resistence in a screw.*

A screw consists of a bolt in the form of a circular cylinder with a uniform projecting *thread* running round its surface. The inclination of the thread at all its points to a plane perpendicular to the axis of the cylinder is always the same.



The distance between any two consecutive threads measured parallel to the axis is called the **pitch** of the screw.

The screw thus formed fits into a hollow nut or fixed support with a corresponding groove on its internal surface in which the ridge of the screw slides, As the thread slides into the groove, the screw can only revolve about its axis and at the same time it can move forward parallel to its length.

If a be the length of the arm at the end of which the effort P is applied and p the pitch of the screw, then in one complete revolution the distance moved by the end of the arm is $2\pi a$ and that moved by the screw parallel to its axis is eaqnal to p.

Let W be the resistance parallel to the axis of the screw.

Then neglecting friction, we have by the principle of work

$$P.2\pi a = W\,p$$

$\therefore$ the mechanical advantage $W/P = 2\pi a/p$.

Ex. 1. What is the pitch of a screw press, if the handle is 2 ft. long and a force of 10 lb wt. applied at each end of the handle cause a pressure of 5000 lb. wt. to be exerted by the press (neglect friction) ? [*Ans* 0·3 in.

Ex. 2. If the thread of screw makes 30 turns in 11 inches what power at the end of a lever 25 in. long will sustain a resistance of 1000 lb ? [*Ans* $2\frac{1}{3}$ lb.]

Ex. 3. A screw press that has a screw of two theads per inch is required to exert a pressure of 2 tons. Assuming that the efficiency is 35 per cent what must be the length of the handle so that the effort does not exceed 40 lb ?

[*Ans.* $25\frac{5}{11}$ in.)

CHAPTER XIII

GRAPHICAL STATICS

13.1. We have already dealt with the analytical method of finding the magnitude, direction and line of action of the resultant and the conditions of equilibrium of a number of co-planar forces. In this chapter we propose to do the same thing graphically and give the graphical methods to the solution of certain problems in Statics.

The particular case o forces acting at a point, say A, was considered in Art. 3·22 and it was shown that if a polygon is drawn so as to have its sides parallel and proportional to the forces, the line joining the first and the final points represents the resultant of the forces in magnitude and direction, the point

of application of the resultant being the known point A at which the forces act.

We shall now consider the more general case of co-planar forces acting on a rigid body.

13·2. *To find graphically, the magnitude, direction and line of action of the resultant of a number of co-planar forces.*

Let P, Q, R, S be the given forces, acting along given lines. (Fig. *i*.)

We take only four forces but the method is a general one and can be applied to any number of forces.

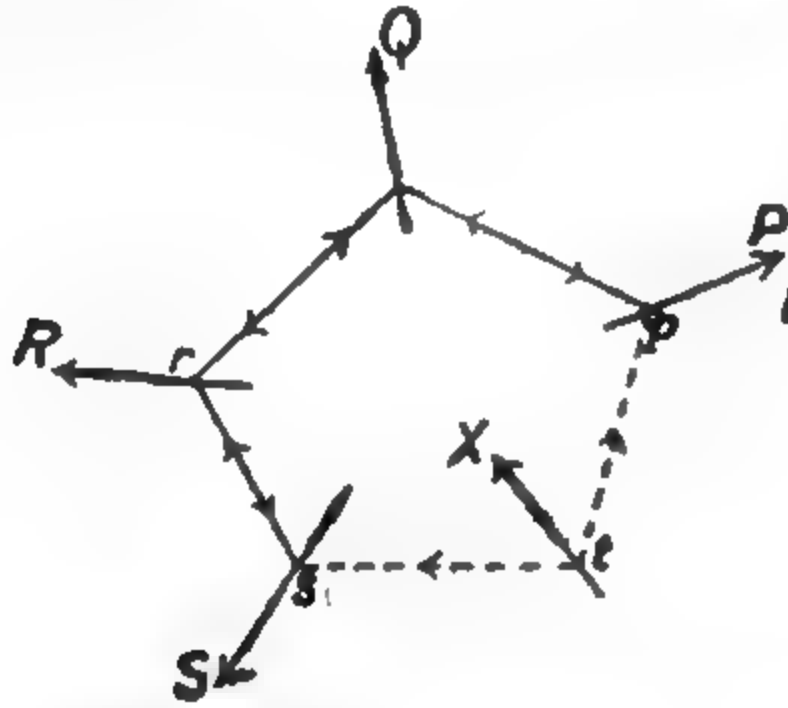


Fig (i)

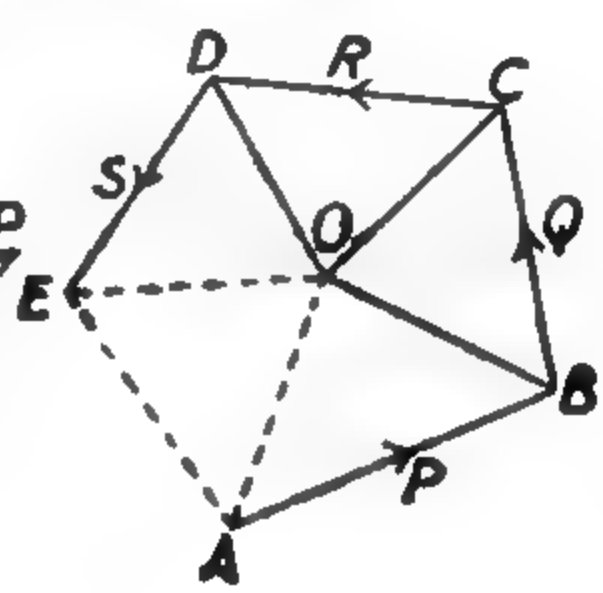


Fig. (ii)

Draw the figure ABCDE (Fig. *ii*) having its sides AB, BC, CD, DE parallel and proportional to P, Q, R, S respectively and join AE.

It will be shown that AE represents the resultant force in *magnitude* and *direction*.

Now take any point O in the plane inside or outside ABCDE but not on its perimeter, and join OA, OB, OC, OD and OE.

Then in Fig (*i*), take any point *p* on the line of action of the force P and draw *pq*, *qr* etc such that

pq is parallel to OB, meeting the line of action of Q in *q*.
qr is parallel to OC, meeting the line of action of R in *r*,
rs is parallel to OD, meeting the line of action of S in *s*,

Lastly, from *s* draw a line parallel to OE and from *p* draw a line parallel to OA meeting in *t*.

Then we now prove that *the resultant force is equal and parallel to AE and acts through t.*

The force P acting at p=AB=AO along tp+OB along qp.
The force Q acting at q=BC=BO along pq+OC along rq.
The force R acting at r=CD=CO along qr+OD along sr.
The force S acting at s=DE=DO along rs+OE along ts.

The forces acting along the lines pq, qr, and rs, being equal and opposite and collinear in action, cancel in pairs, and we are left with only the first force AO along tp and the last force OE along ts, and these forces have a resultant, say X, equal and parallel to AE acting at the point t.

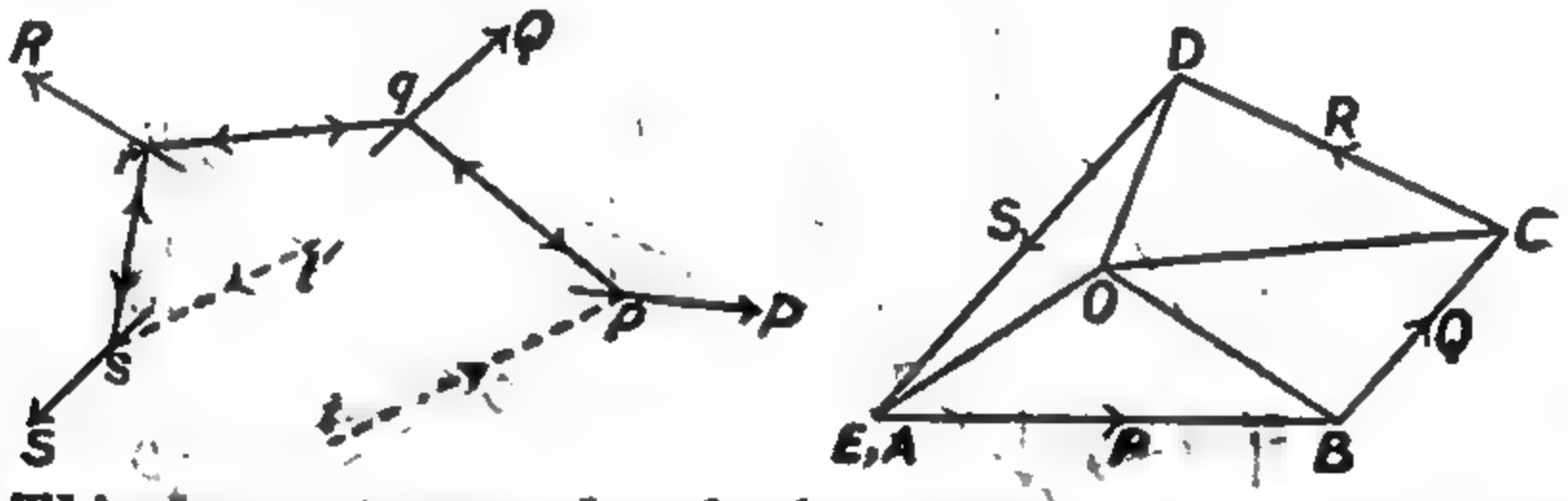
The figure ABCDE is called the **Force Polygon**, the point O is called the **Pole** and the figure $pqrst$ is called a **Funicular or Link Polygon.**

Note. 1. We have assumed that the given forces are acting upon the same rigid body, but if the points p, q, r, s and t were joined by light ropes (or by smoothly jointed light rods) forming the sides of the link polygon, the force X reversed and the forces P, Q, R, S would maintain equilibrium.

This justifies the same funicular (rope) polygon.

Note. 2. It is clear that by taking different positions of the point O and p, the shape of the funicular polygon can be changed in any number of ways but the final point of intersection of the lines pt and st will always lie on the same straight line, the line of action of the resultant X.

13·21. The resultant, a couple. When the first point A of the force polygon coincides with the last point E, *i.e.*, If the polygon is closed, the resultant is zero.



This does not mean that the forces are necessarily in equilibrium.

In fact the forces may reduce to a couple or they may be in equilibrium. The funicular polygon helps us in distinguishing the two cases.

When A and E coincide, the lines pt and st' drawn through p and s parallel to OA and OE will be parallel and the forces acting along them will be equal and opposite, so that the resultant is a couple whose moment is equal to

OA $\times$ distance between the parallel lines pt and st'.

Note. The above discussion shows graphically that a system of coplanar forces, not in equilibrium, reduces either to a single force or to a single couple.

13·22. Conditions of equilibrium. If, however, the lines pt and st' coincide *i.e.* $s'p$ is a straight line, the equal and opposite forces along them cancel and the given forces are evidently in equilibrium. In this case the funicular polygon is also closed.

Thus in order that a system of coplanar forces may be in equilibrium, two conditions must be satisfied :

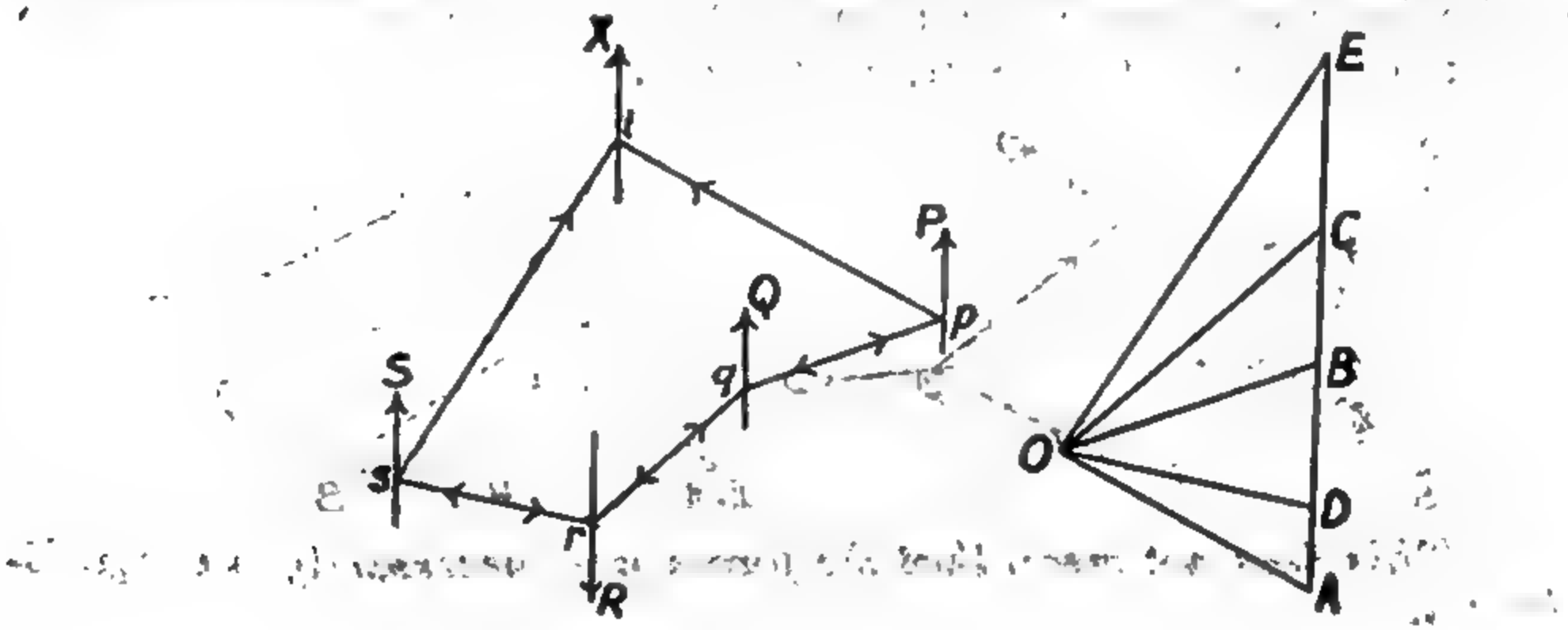
(1) The Force polygon must be closed.

(2) The Funicular polygon must be closed.

These conditions are necessary as well as sufficient.

13·23. Parallel Forces. The same method applies to the particular case when the forces are parallel. In this case the force polygon reduces to a straight line.

Let P, Q, R, S be the parallel forces acting along given lines. Let AB, BC, CD, DE represent these forces in order, on some scale.



Take any point O as the pole and join it to A, B, C, D and E.

Take a point p on the line of action of P and construct the funicular polygon $pqrst$ as in Art. 13.2. From the same proof as given in Art. 13.2, we get the resultant X acting at t.

Example. A light rod AB supported horizontally at two points, has a weight of 5 lb. at A and weights of 4 lb. and 9 lb. are placed on it at distances of 3 and 9 inches from A. If the reaction at one of the supports situated at a distance of 7 inches from A, is 7 lb. Find, graphically, the other reaction in magnitude and position, and verify the result by calculation.

Draw the force and funicular polygons just as done above and measure the horizontal distance of t from the line of action of the force at A.

13 24. *A closed polygon of light rods freely jointed at their extremities is in equilibrium under the action of a given system of forces applied at the joints ; to find the stresses along the rods.*

In Fig. (i) Art. 13.2, take the force X in the reversed direction, so that the forces P, Q, R, S and X form a system in equilibrium and the lines pq, qr, rs, st and tp represent the rods freely jointed at their ends where forces act,

Draw the force polygon ABCDEA having its sides parallel and proportional to the forces P, Q, R, S, X respectively. Since the forces are in equilibrium, this polygon must close.

Now draw AO parallel to tp and BO parallel to pq, meeting in O. Join OC, OD, OE, then OA, OB, OC, OD, OE represent in magnitude and direction the stresses along the rods to which these lines are parallel.

This is proved as follows :—

Let T_1, T_2, T_3, T_4, T_5 be the stresses along the rods pq, qr, rs, st, and tp.

The triangle AOB has its sides parallel to the forces P, T_1 and T_5 which act at the point p and are in equilibrium. Its sides are, therefore, proportional to these forces ; hence the sides BO and OA represent T_1 and T_5 on the same scale that AB represents P.

Again the sides OB, BC of the triangle OBC represent two

of the forces T_1 and Q which act at q, hence OO represents the third force acting at q *i.e.*, T_2.

Similarly DO represents T_3 and EO represents T_4.

Ntote. Whenever a *light* rod is in equilibrium under the action of forces which can act only at its ends then the forces at the ends must act *along* the rod and must be equal and opposite. A rod that is in compression or that is being pushed is called a 'strut' and the one that is being pulled or is in tension is called a **'tie'**.

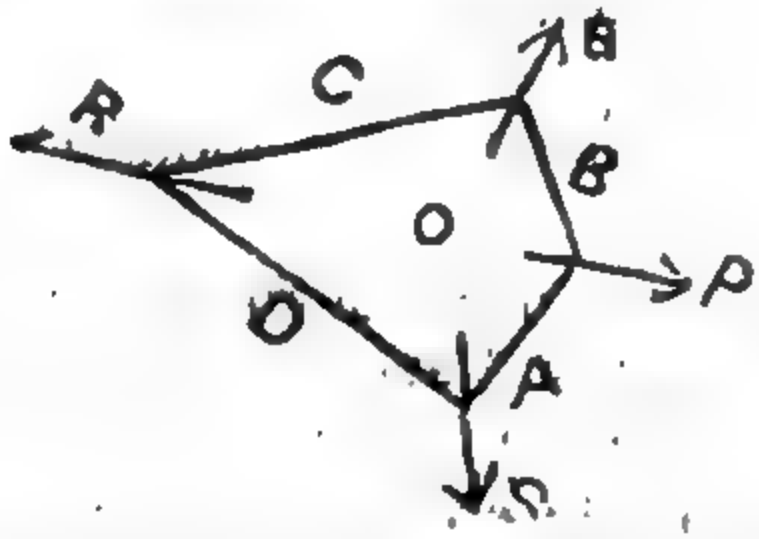
13.241. Reciprocal Figures. It is clear that ABCDE is the force polygon and *pqrst*, a funicular polygon for the forces P, Q, R, S and X.

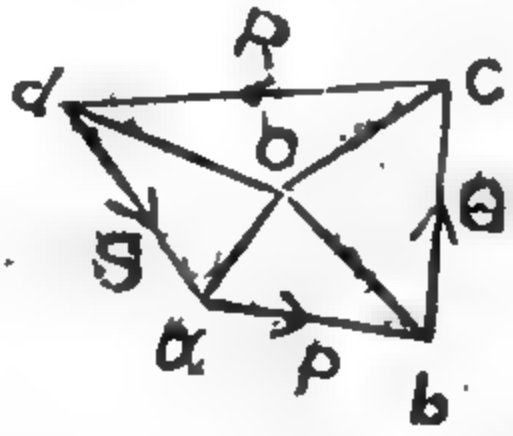
Also if ABCDE is taken to represent a jointed framework of rods acted on by forces OA, OB, OC, OD, OE, then *pqrst* is the force polygon for these forces, ABCDE being the funicular polygon.

Hence either of these two polygons may be taken as the framework of funicular polygon and then the other is the force polygon. For this reason such figures are called *Reciprocal Figures.*

13·3. Bow's Notation.

Let the space between the lines of action of forces P and Q be denoted by B, that between Q and R by C and so on. In the force polygon the force P is represented by the line *ab* while in the funicular polygon the same force is denoted by AB (the same capital letters) *i.e.* by the names of spaces it separates.





Corresponding to a point *o* within the force polygon, we have O, denoting the space within the funicular polygon.

This is known as Bow's or Henrici's notation and has the following advantages :—

(1) The same letters represent a given force in both polygons.

(2) The lines in the funicular polygon drawn parallel to *oa*, *ob*, etc., lie in spaces A, B etc.

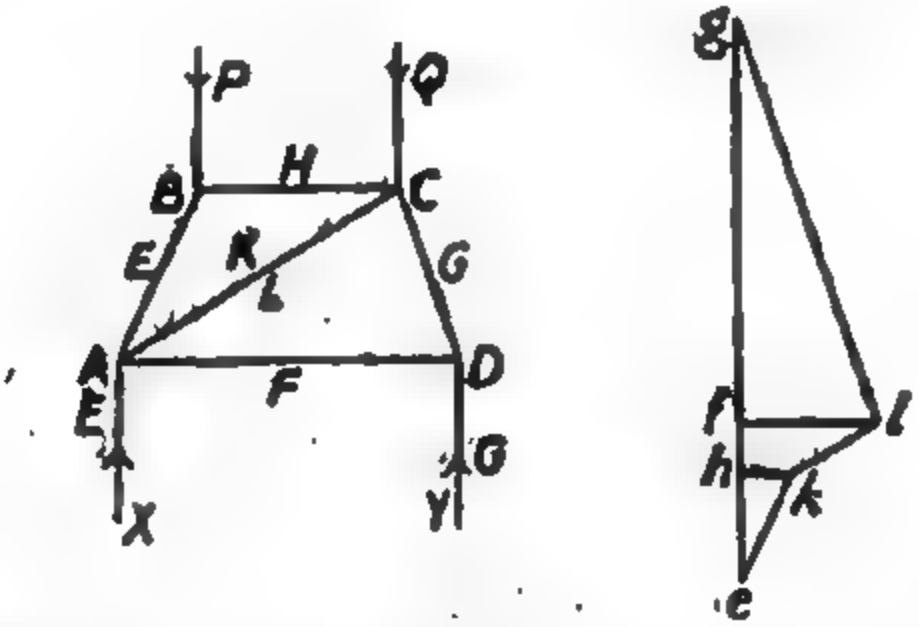
(3) The point of application of any force, sav P, is denoted by AOB, so that the forces meeting at any point correspond to the triangle *aob*, the triangle of forces for the point.

This notation is specially useful in framework problems.

Note. Bow's notation does not apply if there are forces on the rods at other points than the hinges.

Example. *ABCD is a framework of light rods stiffened by a brace AC. It is supported in a vertical plane by supports at A and D, so that AD and BC are horizontal and AB, CD equally inclined at a given angle to AD. If weights P and Q are placed at the hinges B and C, find the stresses in the rods and the thrusts X and Y at A and B.*

Following Bow's notnation areas separated by the lines of the figure are marked as E, F, G, H, K, L. Any force in the figure is named by the letters of the two areas separated by its line of action.



Start with some point say B, one out of the three forces acting at the point being known.

The forces at B are HE (i.e., the known weight P), EK and KH (taking the letters in counter-clockwise direction).

Draw *he* to represent P (on some scale), *ek* parallel to AB and *hk* parallel to BC, meeting in *k*., then *ek* and *kh* represent the force EK and KH, *i.e.*, the stresses in AB and BC.

Now consider the point C; the forces acting at it are GH (*i.e.*, the known weight Q), HK (now known), KL and LG.

Draw *gh* to represent GH or Q and complete the force polygon *ghkl*, by drawing *kl* parallel to AC and *gl* parallel to CD, meeting in *l*.

The sides *kl*, *lg* represent the forces KL, LG *i.e.*, the stresses in AC and CD.

Now take the point D, the forces acting at it are GL (now known), LF and FG. Complete the force triangle *glf* by drawing *lf* parallel to DA, meeting *gh* in *f*. The side *lf* represents the force LF *i.e.*, the stress in AD, and *gf* represents GF *i.e.*, Y.

Now the figure is complete, the forces at A, KE, EF or X, FL and LK being represented by the sides *ke*, *ef*, *fl* and *lk*.

Ex. Given AB=CD=3, BC=2 and AD=4, P=100 lb., Q=50 lb. show that X=87·5, Y=62·5 and that the stresses along AC and AD are 18 2 and 22 lb. Also show that AC, AD are ties while the other three rods are struts.

EXAMPLES XLI

1. AB is a straight line, 3 ft. long; at A and B act (*i*) like, (*ii*) unlike parallel forces equal to 70 and 50 lb. respectively. Find graphically the position of the point where the resultant acts and verify the results by calculation.

2. Parallel forces of 3, 5, 7 and 11 lb. act at points in a straight line at distances of 4 in. from each other. Find the position of their resultant graphically and verify it by calculation.

3. A horizontal beam AB, 18 ft. long has loads 5,11,18 tons at distances of 6, 10, 15 feet from A. Find graphically the supporting forces at the ends. [*Ans.* $11\frac{2}{3}$, $22\frac{7}{3}$ tons.]

4. An equilateral framework ABC formed of light jointed bars, is suspended by means of two vertical strings attached

to A and B so that AB is horizontal. A weight of 15 lb. hangs from C, find the stresses in the bars of the frame.

[*Ans.* $5\sqrt{3}$, $5\sqrt{3}$, $\frac{5}{2}\sqrt{3}$ lb. wt.]

5. ABC is a triangle whose sides AB, BC, CA are respectively 12, 10 and 15 inches long and BD is the perpendicular from B on CA. Find by means of the force and funicular polygon the magnitude and the line of action of the resultant of the following forces :

(*i*) 8 from A to C, (*ii*) 8 from C to B, (*iii*) 3 from B to A and (*iv*) 2 from B to D.

[*Ans.* 2·6 lb. cutting BC and AC produced at distances 19·25″ and 17·6″ from C].

6. A triangular frame of jointed rods ABC, right-angled at A can turn about A in a vertical plane. The side AB is horizontal and the corner C rests against a smooth vertical stop below A. If AB=3 ft., AC=1 ft. and a weight of 50 lb. be hung on at B, find graphically the stresses in the various rods. [*Ans.* 150, 158·115, 50 lb wt.]

7. A uniform beam AB weighing 60 lb, is supported by strings AC and BD, the latter being vertical and the angles CAB and ABD are each 105°. The rod is maintained in this position by a horizontal force X applied at B. Show that the value of X is about 15 lb. wt.

8. A and F are two points joined by a string whose length is greater than AF. At intermediate points B, C, D, E etc. of the string weights are attached, each epual to W. Assuming that each section of the string slopes upward to the right, show by means of a force diagram that the tangents of the inclinations of the successive portions of the string to the horizon are in Arithmetical progression.

9. Three light bars are jointed together to form a triangular frame work ABC in which the angles A and C are each 30°. The framework can turn in a vertical plane about the point B, and is kept in equilibrium with AB horizontal by a weight of 100 lb. hung at C and a vertical force P at A.

Find graphically the magnitudes of the force P and of the stresses in the rods.

[*Ans.* P=50 lb. wt. : Thrusts in AB=86·6 ; in BC=173 ; tension in AC=100.

10. A portion of a Warren girder consists of three equilateral △s ABC, ADC, BCE, the lines AB, DCE, being horizontal, and AB uppermost. It rests on supports at D and E and carries 3 tons at A and 1 ton at B. Find the reactions at the supports and the stresses in the four inclined bars, stating whether they are in tension or compression.

[*Ans.* 2·5 tons at D, 1·5 tons at E. Thrusts in AD=2·88 tons, AC=·58 ton, BE=1·73 tons ; tension in BC=·58 tons.

HINTS FOR SOLUTION

Ex. II.

4. By (2) Art. 2·12, $\tan\theta=\dfrac{2P\sin\alpha}{P+2P\cos\alpha}=\dfrac{4P\sin\alpha}{(P+4)+4P\cos\alpha}$.

$\therefore$ $2P+4P\cos\alpha=P+4+4P\cos\alpha$ *i.e.*, $P=4$.

7. Fig. Art. 2·11, $\angle AOC=\frac{1}{3}\alpha=\angle OCB$ and $\angle COB=\frac{2}{3}\alpha$

$\therefore$ $\dfrac{P}{Q}=\dfrac{BC}{OB}=\dfrac{\sin\frac{2}{3}\alpha}{\sin\frac{1}{3}\alpha}=2\cos\frac{1}{3}\alpha$ *i.e.*, $\alpha=3\cos^{-1}\dfrac{P}{2Q}$.

Now use $\cos\alpha=4\cos^3\frac{1}{3}\alpha-3\cos\frac{1}{3}\alpha$ and then find R.

8. The resultt. makes angles $\alpha-\theta$, $\alpha+\theta$, with the forces

Hence $\dfrac{P-Q}{P+Q}=\dfrac{\sin(\alpha-\theta)}{\sin(\alpha+\theta)}$, find $\dfrac{P}{Q}$.

12. Fig. Art. 2·11, P being fixed in magnitude and direction, A is a fixed pt., AC=OB=Q being fixed in magnitude, locus of C is a circle with centre A.

The inclination of OC with OA is greatest when OC is tangent to the circle.

13. If α is the angle between P and Q and θ that between R and P, then $R\cos\theta=P+Q\cos\alpha=Q$

i.e., $P=Q(1-\cos\alpha)=2Q\sin^2\dfrac{\alpha}{2}$.

Ex III

3. Fig. Art. 2·11, produce OB to B′ making OB′=Q′ and complete the parallelogram OAC′B′, so that OC′=R′. Now the the resultt. of R′ reversed and R *i.e.*, of C′O and OC is C′C which is the difference of OB′ (Q′) and OB (Q).

5. (*i*) Force AP and PB=AB ; AB and BC=AC acting at P. Thus PQ is equal and parallel to AC.

(*ii*) D mid-point of BC ; forces PC and PB=2PD. Now AP+2PD=AP+PD+PD = AD at P+PD = PD+DQ = PQ where Q is on AD such that DQ=AD.

6. P acts along AD ; D being the pt. where Q and R meet. CAD is the triangle of forces. Find the angle A and then the length AD.

Ex. IV.

7. Let AB=2a, α the angle, each portion of the string makes with the vertical and T, the tension, then by symmetry, 2T cos α=15. Also by Hooke's law T=λ (a cosec α—a)/a. α=30° satisfies the equation in α obtained by eliminating T.

Ex. VI.

3. BC slopes upwards. **4.** BC slopes downwards.

8. First consider the equilibrium of A and find the tension T, in AB or AC, by the equation 2T cos 30°=W.

13. Use triangle of forces ABG and ACG. G is mid-pt. of AB.

14. Use triangles of forces. Note that tension throughout the string is the same.

16. P, Q the point on the sphere where the weights rest. Let OP=l_1, then OQ=l—l_1 ; OPC and OQC are $\triangle$s of forces.

Ex. VII.

7. $P=\frac{P}{OA}$. OA, $Q=\frac{Q}{OB}$. OB. Now apply Art. 3.1.

8. AOA′ diameter through A, BPCA′ is a parallelogram.
∴ PA′ bisects BC, say in D.

Now, forces PB and PC$=2$PD$=$PA′,
$\therefore$ forces AP$+$PA′$=$AA′.

9. Forces BH and CH$=2$DH, D is mid-pt. of BC. Now AH$+2$DH$=2$OD$+2$DH$=2$OH ($\because$ AH$=2$OD)

10. The force AX$=\frac{m}{l+m}$. AB and $\frac{l}{l+m}$. AC. etc.

Thus the system is equivalent to three forces proportion to the sides of the triangle and acting at a point

Ex. VIII.

5. Note that the numbers 7, 31—7$=$24 and 25. form a right-angled $\triangle$.

6. Introduce two equal and opposite forces, each equal to 1 lb. along the third side. Forces 2, 6, 10 and 12 along the sides are in equilibrium.

10. Resultant of forces AP, AQ, AR is 3AG, G being the centroid of P, Q, R or forces AP, PB$=$AB etc.

11. Resolve along and perp. to the fifth side of the hexagon.

Ex. IX.

6. Through the mid-pt. G of the rod AB, draw GD parallel to AC, C being the peg. G D is the triangle of forces.

11. The line of action of the weight, the reaction at A both meet the string BD at its mid-pt. O. Draw BF horizontal, meeting the wall in F, then BF$=a$ sin θ and DF$=2$AF $=2a$ cos θ.

$\therefore$ BD$^2=l^2=$BF$^2+$FD$^2=a^2$ sin$^2\,\theta+4a^2$ cos$^2\,\theta$.

14. Resolve vertically and take moments about the end in contact with the wall or proceed geometrically.

Ex. X.

11. Resolve along and perp. to plane and simplifying, obtain

$$P=W\sin\alpha-\mu W\cos\alpha \qquad \ldots\ldots(1)$$

and $Q(\cos\alpha+\mu\sin\alpha)=W(\sin\alpha-\mu\cos\alpha)=P$ $\qquad\ldots\ldots(2)$

Now divide (1) by W and (2) by Q and square and add to eliminate α.

Misc. Ex. I.

3. A_1PA_2, A_3PA_4 equally inclined at $\angle\alpha$ to the radius OP. Draw OC, OD perp. to these chords, then OC=OD, CP=DP,. Forces PA_2, $PA_1=2PC$; PA_4, $PA_3=2PD$;

$2PC$, $2PD=4PC\cos\alpha=4PC\sec\alpha.\cos^2\alpha=4PO\cos^2\alpha$.

4. $\therefore$ $\tan A=OD/AD$, $\tan B=OD/BD$

$\therefore$ AD tan A=BD tan B, hence the resultt. acts along AD and =(tan A+tan B).OD.

5. The resultt. has also to pass thro' the mid-pt. of BC.

7. Resultt. of forces along OB and OC prop. to AC and AB meets BC in D such that AC.BD=AB.CD, *i.e.*, BD/CD =AB/AC, so that AD is the bisector of $\angle$BAC.

11. Let AC, BD be parallel lines, AB third line. Let the force F interest AC in C. Join CB, resolve F into F_1 along AC and F_2 along CB, and then resolve F_2 along DB and AB.

13. A, B fixed pts. On AB construct a segment of a circle containing an angle ACB equal to the given angle.

Let the resultt. of the given forces acting along CA, CB intersect the circle in D. $\therefore$ $\angle$ACB is constant and forces are of given magnitude, $\angle$ACD is const. $\therefore$ D is a fixed point.

14. Let T be tension in AB or CD, T_1 in BC, then by Lami's theorm

$$\frac{T}{\sin 90} = \frac{T_1}{\sin(180-\theta)} = \frac{\lambda}{\sin(90+\theta)}$$

$\therefore$ $T=\lambda\sec\theta$ and by Hooke's law $T=\lambda(AB-l)/l$.

$\therefore$ $AB=l+l\sec\theta$.

Also $T_1=\lambda\tan\theta=\lambda(BC-l)/l$, *i.e.*, $BC=l+l\tan\theta$.

Now $AD=2l=2AB\sin\theta+BC$ gives the desired result.

15. Equate the values of tensions in BC obtained by considering the equilibrium of the weights at B and C.

16. T, the tension in each string, α the angle each makes with the rail, R the vertical reaction on a ring. For one of the rings $T\sin\alpha=R$, $T\cos\alpha=\mu R$, $\therefore$ $\mu=\cot\alpha$,

18. Considering Q which is on the point of motion down the plane of inclination β.

T, tension in string $= Q(\sin\beta - \mu\cos\beta)$

$= Q\sin(\beta-\lambda)/\cos\lambda.$

Similarly considering P, on the point of motion up the other plane.

$T = P\sin(a+\lambda)/\cos\lambda$. Hence $P\sin(a+\lambda) = Q\sin(\beta-\lambda)$...(1)

If x be the required weight which when added to P make the system just on the point of motion in the reversed direction obtain another equation (2) by replacing P by $P+x$ and by changing the sign of λ in (1). Then find x by eliminating P from (1) and (2).

R. Q. I.

1. Resolving horizontally $T_1\sin a = T_2\sin a$, *i e.*, $T_1 = T_2$

By Hooke's law $T_1 = \lambda(c\ \text{cosec}\ a - a)/a = T_2 = \mu(c\ \text{co ec}\ a - b)/b$.

3. Find the length $MN = a/\sqrt{3}$ and apply Ex. 10 Misc. Ex. 1.

4. Let R, R′ make angles θ, θ' with P, then

$R\cos\theta = P + Q\cos a$; $\quad R'\cos\theta' = P' + Q'\cos a$

$R\sin\theta = Q\sin a$; $\quad R'\sin\theta' = Q'\sin a$.

$\therefore RR'\cos(\theta-\theta') = (P+Q\cos a)(P'+Q'\cos a) + QQ'\sin^2 a.$

5. Obtain $\tan\theta$, $\tan\theta'$ and put $\theta' = \theta + a$.

6. $R' = k^2\cos^2 A + k^2\cos^2 B + 2k^2\cos A\cos B\cos C$

$= k^2[\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C]$

$- k\cos^2 C = k^2 - k^2\cos^2 C = k^2\sin^2 C.$

7. Force $AB = AO + OB$, $2BC = 2BO + 2OC$,

$3CD = CO + 3OD$, etc.

Thus $AB + 2BC + 3CD + \ldots = AO + BO + CO + \ldots + n.OA$

$= BO + CO + \ldots + (n-1)OA$

$= (n-1)OA.$ Art. 3.21.

12. Let ABCDE be the polygon. Let P_1, P_2, P_3, P_4, P_5 be the forces corresponding to the sides AB, BC etc., then by Ex. (*iii*) Art. 2.3 , the resultt. of P_1, P_2 is a force acting at the

mid-pt. of AC perp. and prop to AC. Again this force and P_3=a force at the mid-pt, of AD perp. and prop. to it. Now this force, P_4 and P_5 are in equilibrium. The argument can be extended to the case of a polygon of any number of sides.

Ex. XI.

8. It implies that $P<Q<2P$.

11. (*ii*) Let AO meet BC in D and let P and Q be the required forces at B and C, then P. BD=Q.DC.

$$\therefore \frac{P}{DC}=\frac{Q}{BD}=\frac{P+Q}{BC}=\frac{R}{a}$$

But $$\frac{BD}{DC}=\frac{\sin \angle BOD}{\sin \angle COD}=\frac{\sin (\pi-2C)}{\sin (\pi-2B)}=\frac{\sin 2C}{\sin 2B}$$

$$\therefore \frac{BD}{\sin 2C}=\frac{DC}{\sin 2B}=\frac{BD+DC}{\sin 2B+\sin 2C}=\frac{a}{2 \sin A \cos (B-C)} \text{ etc.}$$

Ex. XII.

3. The vertical line midway between the strings passes thro' the C. G. of the cone.

Ex. XIV.

4. Resolve along and perp. to AC,

9. Resolve along and perp. to AB.

11. Since the moments about A and C are equal and opposite, the line of action of the force must meet AC in its mid-pt. D. Again since the moments about B is twice that about A, the force must meet BA produced at E such that BE=2AE. Now AD, AE being known, find $\angle AED$ and the length of the perp from A on DE.

Ex. XV.

7. Take moments about C and resolve forces along the rod.

8. Find $\angle ACB$ by equating the values of tension obtained by Hooke's law and by taking moments about A.

9. Let $\theta=\angle BDA=\angle BAD$; T, tension in the string ; $AB=a$; W, the weight of the lamina ; G, its C. G.

Resolve vertically, T cos $\theta = W$. Take moments about A.

W.AG sin $(30° + \theta) =$ T.AD sin θ, where AG $= a/\sqrt{3}$, AD $= 2a \cos \theta$.

Now eliminate T and obtain tan $\theta = 1/3\sqrt{3}$.

10. Find θ, the inclination of the string to the vertical by equating the values of the tension obtained by resolving vertically and by taking moments about the corner in contact with the wall. Proceed as in Ex. 9.

Ex. XVI.

6. In order that CD may be the smallest, the support D must coincide with the centre of the weights, so that the system is just on the point of turning about D.

(*ii*) Let the portion cut off be x, then the weight of the rod $12-x$ acts at a distance $\frac{1}{2}(12-x)$ ft. from either end. Now x is the greatest when the system is on the point of turning about D. Take moments about D.

Ex. XVII.

1. Take moments about one of the sides of the triangle in case the forces do not act in the plane of the triangle.

10. While calculating the pressure of the table on the prism do not forget to take the vertical component of the thrust.

Misc. Ex. II.

2. Take moments about L, Q.LM sin $\angle M = $ R.LN sin $\angle N$

$$\therefore \frac{Q}{R} = \frac{LN.\sin \angle N}{LM.\sin \angle M} = \frac{LN.OM}{LM.ON} \text{ etc.}$$

3. Take moments about one of the vertices and use property of a cyclic quadrilateral that the sines of the opposite angles are the same.

4. Take moments about the top of the obstacle.

7. Take moments about the vertices of the pedal triangle.

8. Take moments about the sides of the triangle.

9. The plate being horizontal, the extension in each string is the same.

$\therefore$ tension T_1 in AP or CR : tension T_2 in BQ or DS $=\lambda : \mu$

Also $W=2T_1+2T_2$.

10. Let GL, G being below AB, be perp. on AB, and let vertical thro' G meet AB in C. Taking moments about B,

$P \cdot AB \cos a = W.BC \cos a$, *i.e.*, $P.a = W(BL-LC)$

$= W(BL-GL \tan a)$

Similarly $Q.a = W(BL-GL \tan \beta)$ Subtract and find GL.

11. See solved Ex. 2 Art. 2·5.

14. Taking moments about C, $F=6W/13$. Friction at C =horizontal component of $F=30\ W/169$.

C will not slip if $F<\mu R$. Find the normal reaction R carefully.

16. The resultt. of friction F and wt. W meets P in O, a point different from the centre. $\therefore$ the normal reaction must also pass thro' O. If $\mu>1$, it is easy to prove that the cube topples before it slides.

17. (*i*) In case the bar slips at A, C is fixed, take moments of the horizontal forces (P and friction at A) about C.

Similarly find the value of P in the second case.

R. Q. II.

3. The line of action of p is tangent to the plate. Take moments about the centre.

4. See hint for Ex. 10. Misc. Ex. II. (given above)

5. Let O be the centre where nW acts (along OB) and C, the point on the rim where W is attached. The resultt, $(n+1)W$ acts at G on OC such that $OG = OC/(n+1) = a/(n+1)$, AG, being vertical, is parallel to OB. Similarly OA is parallel to BG. Since OA=OB, OAGB is a rhombus and its diagonals OG, AB bisect each other at right angles at E.

$\therefore \angle AOB = 2\angle AOE$.

$\therefore \sec \angle AOE = OA/OE = OA/\frac{1}{2}OG = 2(n+1)$.

6. Take moments about AB, BC and AC.

7. G. the centre of the plank which is in contact with the highest point of the cylinder in the first position, C the point of contact in the displaced position, $GC=a\theta$, find the sum of the moments of the three weights about C.

8. Resolve along and perp. to the rod in its displaced position and take moments about the point of contact.

9. Prove that $AC/CB=M_1/M_2$. The line of action of R is the common tangent to the circles drawn with centres A, B and radii M_1/R, M_2/R respectively.

10. The bisector of the angle between the lines of action of P and Q, the direction of one of them being reversed.

Ex. XVIII.

1. Resolve the fifth force along AB and AD.

2. By Art. 5·51, the moment of the resultt. couple = sum of the moments of all the forces at any pt. say $A=Qb-Pa$.

6. Let AD be the bisector of $\angle A$, AL, the altitude. In the second position the resultt. (=25 lb.) of forces at B and C acts at D perp. to BC and forms a couple with the force of 25 lb. acting at A along AL.

Ex. XIX

1. Introduce equal and opposite forces each represented by BD along BD.

2. Take moments about the centroid.

3. Since $BP : PC=(1+\lambda) : (1-\lambda)$

$\therefore \frac{1}{2}(1+\lambda)AB+\frac{1}{2}(1-\lambda)AC \equiv AP$ (Art. 3·1).

Similary force $BQ \equiv \frac{1}{2}(1+\lambda)BC+\frac{1}{2}(1-\lambda)BA$ etc.

5. Couple $2\triangle OBC \equiv$ forces OB, BC, CO etc.

6. The force Ka at A′ is equal to force Ka along BC + a couple of moment 2K times the area A′BC, etc.

Ex. XX

9. At B, the system $\equiv$ the same force R at A and couple G, G being equal to the moments of R about B.

10. Take moments about F, A and E.

12. By (2) Art. 5·75, $G_1=G-Y$, $G_2=G+2X$, $G_3=G+3X-2Y$. Find X and Y and tan $\theta=Y/X$.

Ex. XXI

3. The resultt. is a force acting along a line parallel to the given line. If, however, the moment is zero, either the resultt is a force acting along the given line, or the system is in equilibrium.

5. Find G_2, G_3 by taking moments about B and C.

6. Take A(0, 0), B(a, 0), C(x_1, y_1) and let the two systems reduce to X, Y, G and X′, Y′, G′. Equating moments of the two systems about A, we get $G=G'$, again equating the moments about B, we get $G-aY=G'-aY'$ which gives $Y=Y'$. Now do the same thing for C and get $X=X'$. Hence the two systems have the same resultt.

7. Reduce the system to a force R at the given point O and a couple G. Now replace the couple by two forces, one acting along the given line and the other equal, parallel and opposite force acting at O. Finally combine the two forces at O into one force.

8. Take A as origin AB($=a$), X=axis. Let the system be reduced to X, Y and G at A. Replace G by two equal opposite and parallel forces, F, acting at A and B in a given direction. X, Y and F at A can be combined into a single force.

9. Resolve along and perp. to AO, BO, CO. The components $Ka\cos\theta$, $Kb\cos\theta$, $Kc\cos\theta$ along them are in equilibrium. [Ex. (*ii*) Art. 2.32]. Now take moments about any vertex. Since the sum of the moments about A, B, C are equal the resultant is a couple. [Art 5·77 (*b*)].

R. Q. III.

1. Apply Art. 5·7.

2. Introduce two equal and opposite forces along the remaining side, the forces being prop. to it and apply Art. 5·7.

3. Take moments about the line of hinges.

9. Resolving along and perp. to BC, obtain slope of the first resultt.

$=(Y_1 \sin C - Z_1 \sin B)/(X_1 - Y_1 \cos C - Z_1 \cos B)$,
equate it to the slope of the second resultt.

Ex. XXII

2. Take moments about two opposite vertices.

4. (*a*) Equate to zero the sum of the resolved parts of the forces along AB, BC and then take moments about the centre.

4. (*b*) Resolve along and perp. to AB.

5. Equate to zero the sum of the resolved parts of the forces in directions perp. to the lines of action of *b* and *c*. Take moments about the point of intersection of *e* and *f*.

6. Take moments about the centre O, and resolve along lines through O, parallel and perp. to a side.

7. Take moments about A, B, C.

8. Take moments about two of the vertices, using the fact that angle between a tangent and the chord through the point of contact is equal to the angle in the alternate segment.

12. To obtain the first result take moments about O. To obtain the second result, take moments about A and write the result in the form

$$\frac{P}{\quad} \times 2\triangle ABC + R'.d \sin 2B - Q'.d \sin 2C = 0,$$

d being the radius.

Obtain two similar equations by taking moments B and C, multiply the equations by P′, Q′ and R′ and add.

Ex. XXIII

2. Take moments about the centre.

3. Resolve along the plane.

4. The horizontal reaction at A, the tension along the string and the vertical line through G, the C. G. must meet at a point, say O. Take moments about A and resolve in horizontal and vertical directions.

$$OA = AG \cos \angle OAG = AG \sin (\angle GAB + \theta).$$

Also $2AG \cos \angle GAB = AB$

and $$\angle GAB = \tfrac{1}{2}\left(\pi - \frac{2\pi}{n}\right) = \tfrac{1}{2}\pi - \frac{\pi}{n}.$$

Ex. XXIV.

8. Express the result of Art. 6·4 as a quadratic in $\cos\theta$ and show that θ is greatest when the length of the rod is least *i.e.*, when its other end is just on the rim.

10 OC is perp. to AC and AO, OC make the same angle θ with the vertical OG (why ?). Let $OC = x$, then $AO = l - x$ and $(l-x)\cos 2\theta = x$. Also from $\triangle OAG$,

$OA/\sin \angle AGO = AG/\sin \angle AOG$ *i.e.*, $(l-x)\sin\theta = \tfrac{1}{2}a\cos\theta$.

Ex. XXV.

2. Take moments about B and resolve along the plane.

3. Take X, Y for the components of reaction, resolve in horizontal and vertical directions, and take moments about A.

5. Tension along BC and in the vertical portion of the string $= W/10$.

9. Take moments about the mid-pt. of AB.

10. Take moments about the lower hinge.

11. C centre of bowl, D its edge touching the wall. Reaction at A along AC, horizontal reaction at B, and the line of action of the weight meet in O, θ the angle CO makes with the vertical, prove $\tan\theta = \tfrac{1}{2}\tan 30°$ and equate the sum of the horizontal projections of AC and CD to that of AB.

13. For P to be least it must act at D, take moments about A.

Ex. XXVI.

5. Resolve vertically for both beams and obtain reaction at B or $C = W$. Take moments about A for one beam AB. Let D, E mid-pts. of AB, AC, $\angle ACD = \alpha$, AL perp. from A on CD. $= AC \sin\alpha$, From $\triangle ACD$, $CD \sin\alpha = AD \sin \angle CAD = \tfrac{1}{2}AC \sin 2\theta$ and $CD^2 = AC^2 + AB^2 + 2AC\,AD \cos(180° - 2\theta)$ etc.

7. Take X, Y components of reaction at B equal and opposite on the two rods. Take moments about A for AB and about C for BC.

10. Take moments about A for both and also for the rod.

Ex. XXVII

19. The rod AB$=2a$ is greater than AD$=r$ cot θ ; D being the point of contact with the cylinder.

Take moments about A and resolve in horizontal and vertical directions.

20. Let F be the force of friction and R, the normal reaction at A. Take moments about A and resolve in horizontal and vertical directions to find F and R. Apply condition F$<\mu$R.

21. Take AC downwards ; take moments about C and resolve in horizontal and vertical directions.

Misc. Ex. III. A.

3. The couple=moment of W about the point where the reactions of the planes meet.

4. Resolve vertically for the whole system and take moments about A for AB or AC.

5. Resolve vertically for the whole system and also for the rod RC, and horizontally for AB or CD. taking X. Y for the components of the reaction at B or C in these directions.

6. Show that AC$=b\sec(\theta-a)$, resolve along the plane and take moments about A.

7. Let R be the reaction at C, components of reaction at B on BC are $\frac{1}{2}$ W upwards and R towards the wall ; and on AB, the same reactions in opposite sense. Take moments about B for BC and about A for AB. W is the weight of AB.

8. Read $\sqrt{3}l/2$ for $\sqrt{2}l/3$ in the third line.

9. Resolve along the plane for the two spheres separately.

11. Resolve vertically taking the three cylinders together and also for the upper cylinder. Also resolve horizontally for one of the lower cylinders.

12. Take moments about the axis of the shaft and resolve along the plane.

13. Take moments about A, B and C, the perp. from A on BG being written as $2\triangle$ ABG/BG$=2\triangle$ ABC/3BG etc.; multiply the three results by P/AG, Q/BG, R'/CG and add.

14. The equality of the first three terms follows at once from Lami's theorem. For the rest, let R_1, the normal reaction at A, w_1 the weight of the rod AV; take moments about V of R_1, F_1 and w_1 for this rod and similarly for others and add.

Misc Ex. III. B.

6. The vertical line through G must pass thro' the point of contact with the plane.

8. The vertical through A the pt. of contact must pass thro' G, so that $\angle OAG=\alpha$. Also since G must be at a distance c from O, two such positions G_1, G_2 are possible and thus it is possible for the sphere to be turned thro' $\angle G_1OG_2=2\theta$. OL perp. from O on $G_1G_2=c\cos\theta=a\sin\alpha$.

9. The vertical thro' D one of the extreme positions passes thro' G and makes $\angle\lambda$ with the normal OD. $\angle OGD=180°-\beta$ and from $\triangle$ OGD. OG/sin λ=OD/sin β. Substitute the value of OG as given in Ex. Art. 8 6.

13. The horizontal reaction of the wire is towards the centre O at the lower point A and away from O at the higher pt. B. Take moments about A and apply the condition that $F<\mu R$.

14. ABCD represents the drawer, the front face $AB=2a$. Due to the pull applied at a distance $a+x$ from A, there will be reaction R at A in the direction AB and another reaction S at the opposite corner at the back in the opposite sense. The frictions μR, μS oppose motion. Prove R=S; take moments about A and apply condition $P>\mu(R+S)$.

15. In the new position depth of A below the pulley, $P=h-l\sin\theta$ and its distance from PN, the vertical thro' $P=l-l\cos\theta$, hence the angle the string makes with PN is known.

Take moments about A and resolve in horizontal and vertical directions.

16. Proceed as in Art. 7.41.

17. Find F_1, F_2, the two values of the force which are just sufficient to rotate the rod and to make it slide. Apply the condition $F_1 > F_2$ or $F_1 < F_2$.

18. Take moments about A and resolve along and perp. to the plane. (Compare with Q. 6 Misc. Ex. III. A).

19. Take moments about A and resolve in horizontal and vertical directions for the whole system.

20. Take moments about the pt. of contact of the locked wheel with the rails and resolve along and perp to the rails regarding friction at wheels that are free to rotate as negligible.

21. Show that pressure at B $= \frac{3}{4}W$ and at C $= \frac{1}{4}W$ Find P_1, the force applied at A when slipping takes place at B by taking moments about C. Similarly find P_2, the force for slipping to take place at C. Apply the condition $P_1 < P_2$.

23. The rod is clearly least when one end is just under the lower peg and it is on the point of slipping.

24. The resultant reaction passes thro' the point where lines of action of the given force and the weight of the rod meet and is equally inclined to these lines.

25. It is assumed that the smoother body is the lower one Consider the equilibrium of each separately.

27. Let the rod be on the point of slipping when the weight is placed at a distance x from one end A. Find x by taking moments about A and resolving along and perp. to a plane. Again find a similar distance y when the rod is about to slip in the opposite sense. This can be written down at once by merely changing the sign of λ in the value of x. The required length is $y \sim x$.

28. Let P(AP $= x$) be the point of contact of the rod AB with the peg when the frame is about to slip. Then the vertical thro' P will pass thro' O the centre of the figure and will make $\angle\ \lambda$ with the reaction at P perp. to AB. Find x by considering $\triangle$ APO. Find similar distance y for the slipping to take place in the opposite sense by changing the sign of λ ; $y - x$ is the required distance.

29. For the least angle β, the lower end B of the rod is on the point of slipping down the side of the tumbler, and the point of contact with the rim inwards. Take moments about A, resolve horizontally and vertically, obtain a result involving β and λ. For the greatest angle α, the rod is about to slip in the opposite sense, change the sign of λ and replace β by α.

30. The tangent of the angle ϕ between the normal at P($a\cos\theta$, $b\sin\theta$), the point of contact of the wire with the peg, and the line joining P to the centre $=(a^2-b^2)\sin 2\theta/2ab$. In order that the wire may not slip at all points $\tan\lambda$ must be greater than the greatest value of $\tan\phi$.

32. For a cycloid $s=4a\sin\psi$ and $s^2=8ay$.

Ex. XXIX.

13. Take moments about the sides, show that $x=\frac{1}{2}(b+c)/2a$ etc.

14. Take moments about BC, using AD$=c$ sin B and CD$=b$ cos C.

15. $\triangle ABC/\triangle A'B'C'=l^2/x^2$ (why ?). Take moments about D.

18. AG is perp. to BG (why ?).

22. C. G. of the border of mass m is the incentre of the triangle formed by the middle points of the sides. Take moments about two of the sides.

23. Take moments about the side.

Ex. XXX.

9. Take $\angle AOB=2\alpha$, show that $AB/OG_1=3\alpha$ and so on.

18. Take AB as axis of x ; its mid-pt. as origin. Find the co-ordinates of the C. G. of the whole stick and slope of the line CG (which is vertical).

Misc. Ex. IV.

1. (*i*) Apply Arts. 3·1 and 3·21.

(*ii*) $\lambda AA'=\lambda AG+\lambda GG'+\lambda G'A'$.

2. (*i*) $\lambda_1 OA=\lambda_1 OD+\lambda_1 DA$ etc.

(*ii*) They intersect at D, take moments about it.

5. Reaction at C is vertical (why ?). Take moments about B in both cases.

13. Replace the triangle by three equal particles at the vertices.

14 Replace $\triangle$DEF by three equal particles at D, E and F ; take moments about BC. Distance of E from BC=CE sin C $=ab$ sin C/$(a+c)=2\triangle/(a+c)$ and that of F$=2\triangle/(a+b)$.

Ex XXXI.

14. For values of x and y in terms of ψ see (5) and (6) Art. 10·13.

15. Find $\bar{y}/\bar{x}$ without evaluating the integrals in the denominators of $\bar{x}$ and $\bar{y}$ as they cancel out by division.

17. O, the centre ; take OPQ making $\angle\theta$ with OC intersecting the circle in P and Q. First integrate with respect to r, taking its limit from OP$=r_1$ to OQ$=r_2$. Express r_1 and r_2 in terms of θ. The integrals finally cancel and the result follows at once.

18. Consider the sector as made up of a sement and a cone on the same base The condition is satisfied only when the C. G. of the sector lies in its edge *i.e.*, it is at a distance a cos α from the ceutre of the sphere.

Ex. XXXIII.

5. Express PQ and RS in terms of the sides and an angle of the parallelogram.

14. Express the depths of the rings below the focus S, in terms of their distances r_1, r_2 from S and apply the condition that $r_1+r_2=$const.

Ex. XXXIV.

7. O, the centre of the sphere ; A, joint of the rods, $2l$ length of each rod, θ angle made by OA with the vertical in the displaced position. Show that $\bar{\ }=$OGcos θ, where OG$=r$ cosec $\frac{1}{2}\alpha-l$ cos $\frac{1}{2}\alpha$.

11. Pot.Energy of the elastic string=work done in extending it. See Art. 9·12. (Compare Ex. (*iv*) P. 230.)

15. The Virtual work for a further extension dp of a string $=kp^n dp-\lambda\frac{p-a}{a}dp$.

19. Equate the work done by the force to the pot. energy acquired by the string.

Ex. XXXV.

11. The ends of the chain reach the directrix of the catenary.

14. $\frac{1}{2}$ W$=ws=wc\sinh\frac{x}{c}$, x being the abscissa of D, Fig. Ex.(*iv*)Art.10·4 $\frac{1}{2}$W$+wl=w(s+l)+wc\sinh\frac{x+a}{c}$. Eliminate x

21. The other end of the chain must reach the directrix of the catenary. z, the whole length of the string$=y$ the vertical part$+2s$, the curved part$=c\cosh\frac{a}{c}+2c\sinh\frac{a}{c}$. The result follows from $dz/dc=0$.

23. Complete the catenary and apply the formulae :—

$y^2=c^2+s^2$, $y=c\cosh\frac{x}{c}$, $s=c\sinh\frac{x}{c}$ and T$=wy$.

Ex. XXXVI.

3. W is the resultant of the tensions in the two parts of the string, not in contact with the pulley.

4. Equate the horizontal components of the tensions in the two parts of the string.

9. Find the tension at the end of the horizontal diameter, taking the tension at the highest point to be zero.

Ex. XXXVIII.

4. 2^6P$+$P$=$W$=13\times14$ lb.

5. 2^3P$=$W$+w_1+2w_2+4w_3=91+5+2\times4+4\times2=112$. lbs.
Thrust$=$weight of the man$-$P.

6. Stress$+$P$=$total weight$=$W$+4w$.

9. Here $n=8$. **18.** Take moments about B.

Ex. XXXIX.

9. Due to the action of P, the normal reactions and consequently the frictions at the two points of contact of the shaft with a groove are not equal.

Ex. XL.

10. Let the fulcrum be at a distance x from the middle point of the beam, then

$W_1(a-x)=W(a+x)$ and $W_2(a+x)=W(a-x)$; find x.

Ex. XLI.

9. Draw $\triangle$ABC with AB horizontal and C above AB.

Draw a vertical line *ab* to represent 100 lb., on some scale. Draw *bc* parallel to BC and *ac* parallel to AC, to meet in *c*, then *abc* is the triangle of forces for the point C, *ab* representing 100 lb., *bc*, the *thrust* on the rod BC (the point C being pushed away from B) and *ca*, the *tie* in AC, (the point C being pulled towards A in this case).

Now draw *cd* parallel to BA, *cda* being the triangle of forces for the point A, *da* representing P.

It may be noted that *cbd*, the triangle of forces for the point B shows that the reaction at B, represented by *bd* is vertical and equal to $100+$P. This fact is otherwise obvious.

END